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*Keywords*: International transmission of inflation, dynamic general equilibrium, no-Ponzi-Game condition, monetary policy, fiscal policy



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**Abstract**

This paper uses the Neumeyer-Yano's monetary dynamic general equilibrium model to investigate the inter-connectivity of the world economy through bond holding beyond national borders. The possibility that unexpected inflation in one country transmits to another is demonstrated within a framework in which the nominal exchange rate is flexibly determined so that the purchasing power parity holds. Deflation can also be imported through the same channel. Whether inflation or deflation diffuses internationally depends on the level of fiscal deficit. Although a country may suffer from monetary disturbances from abroad, each country can completely defend itself by implementing appropriate fiscal policies.

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# 1 Introduction

Historically, worldwide inflation synchronized among several countries has often been observed. An outstanding and somewhat notorious example was seen in western countries during the first half of the 1970's. With today's rapidly globalizing economic environment, an international inflation (or deflation) spiral may precipitate a possible crisis for the world economy. To understand the transmission mechanism, several studies have been conducted. However, many of the existing theoretical studies have highlighted fixed exchange rate regimes by using short-run macroeconomic models (see, e.g., Turnovsky and Kaspura [1974], Kingston and Turnovsky [1978]).

The main purpose of this paper is to reveal a channel through which foreign inflation (or deflation) transmits to a domestic economy even under a flexible exchange rate regime, from the standpoint of neoclassical long-run theory. To that end, particular focus is placed on an upper bound of national debt that is compatible with a dynamic general equilibrium (DGE) and a no Ponzie-game (NPG) condition for the government as introduced into monetary DGE analysis by Neumeyer and Yano [1995]. This condition for the government is logically equivalent with the repayability of financial liabilities over an infinite time horizon. The present paper reveals how the upper bound is influenced by the monetary policies of both countries.

Findings worth noting are as follows. First, a level of fiscal deficit at each period plays a crucial role to determine whether inflation or deflation is imported. Under a low level of fiscal deficit, inflation is easily transmitted. The foreign government's expanding monetary policy negatively affects the domestic government's financial standing, while such a policy by the home country's government improves it. When a foreign government expands monetary policy, a domestic government has a strong incentive to also expand, resulting in inflation in both countries. Deflation is also imported through the same channel if the fiscal deficit at each period is relatively heavy.

Second, a domestic government can employ fiscal policy as a precaution against the potential risk of monetary disturbances from abroad. By conducting fiscal policy to constantly yield a budget surplus, each country can escape the risk of imported inflation or deflation. More precisely, the upper bound, which is derived as the NPG condition along an equilibrium path, is not negatively affected by foreign inflation under a policy regime with a budget surplus, and foreign deflation is not incentive-compatible for the foreign government.

Third, this transmission channel works even under a flexible exchange rate system and in the long-run in the sense that purchasing power parity (PPP) robustly holds for each period. In such a situation, there seems to be a common intuition among economists and policy makers that monetary phenomena such as inflation are hard to diffuse internationally. A nominal exchange rate can adjust speedily, so the real domestic economy is completely isolated from monetary disturbances occurring abroad. This study uses a framework with a flexible exchange rate and PPP, and demonstrates that inflation rates among several countries can be liaised beyond national borders through intentions of the governments to control the national debts at repayable levels.

Finally, the analyses presented in this paper fully count in an interaction among agents that works not only internationally but also inter-temporally, by appropriately employing long-run DGE theory. It is often said that a serious defect of the traditional short-run macroeconomic model is the lack of consideration of an agent's dynamic optimization and inter-temporal transaction through markets. This study, using the DGE model of Neumeyer and Yano [1995], comprehensively traces market repercussions in a satisfactory way.

This issue has already been discussed by many authors. Casas [1977] highlights the flexible exchange rate case of Turnovsky-Kaspura's model and concludes that a domestic economy is influenced by foreign inflation under a set of more plausible assumptions than those adopted by Turnovsky and Kaspura. However, his result was obtained within a framework of a short-run macro-model. Kolodko [1987] studies the problem from the viewpoint of political economics with special attention to exchange rates, money, capital markets and trade balances. Jeong and Lee [2001] and Yang et al. [2006] empirically explore transmission patterns among G-7 countries. Andreas [2007] conducts a Markov-switching analysis for European countries.

The following studies inspired the present research. Neumeyer and Yano [1995] investigate the world-wide effect of unexpected policy changes. However, the inflation transmission mechanism through the upper bound for a fiscal policy to be sustainable is not investigated in their paper. Kondo [2007] explicitly derives the limitation level for a government borrowing by using a closed economy version of Neumeyer-Yano's model. Although, in Kondo's paper, how the limitation level depends on monetary policy, fiscal policy and primary balance etc is reported, the interdependence of a policy regime across national borders is not explored. Fukuda and Teruyama [1994] and

Ihori et al. [2001] also focus on the NPG condition, and conduct empirical tests about whether or not the Japanese national debt is sustainable. A series of papers, Yano [1990], Nishimura and Yano [1993], Yano [2001], Nishimura et al. [2006] and Been-Lon Chen et al. [2008] investigate the inter-linkage of the world economy within open DGE frameworks. The focal point of these studies, however, is on turnpike properties, nonlinear dynamics and the indeterminacy of equilibria.

This paper is structured as follows. Section 2 offers a base-line model. An equilibrium path is presented in Section 3. Section 4 shows a set of conditions to which the parameters in my model must be subjected. The NPG condition, among others, is highlighted. Section 5 analyzes the upper bound, and demonstrates the possibility of the international transmission of inflation rates. *The appendix provides detailed derivation processes of crucial lemmas. The appendix is written not for publication but for only referee-process.*

# 2 Neumeyer-Yano Model

Think of an open dynamic economy with two countries, a home country $H$ and a foreign one $F$, and with an infinite time horizon. The time structure is discretely indexed by $t = -1, 0, 1, \cdots$. The period between the time points $t-1$ and $t$ is called the period $t$.

There are a representative consumer and government in each country. They transact a consumption good, money and bond at each period. Denote by $p_t$ the price of the consumption good in the home country at the period $t$. The price in the foreign country is denoted by $p_t^*$. As adopted in the literature, variables in the foreign country are distinguished from those of the home country with the superscript $*$. Let $e_t$ be the nominal exchange rate. As discussed in the introduction, this study exclusively analyzes an economy in the long-run, where purchasing power parity (PPP) holds:

$$p_t = e_t p_t^*, \quad t = 0, 1, \cdots. \tag{1}$$

Money helps consumers to transact with each other. A unit of a bond generates an interest rate $1 + i_t$ for the period $t$ to $t+1$. Bonds are traded beyond national borders without transaction costs, and thus interest rate parity holds:

$$1 + i_t = (1 + i_t^*) \frac{e_{t+1}}{e_t}, \quad t = 0, 1, \cdots. \tag{2}$$

The government imposes tax $\tau_t$ on the consumer in a lump-sum manner, and spends it $g_t$ at each period. Both the tax $\tau_t$ and the spending $g_t$ are in consumption good form. The government supplies money $M_t^g$ and bonds $B_t^g$ to consumers through markets. Note that $B_t^g > 0$ means that the government is a debtor, and $B_t^g < 0$ implies that the government is a creditor.

The consumer in the home country initially holds money $M_{-1}$, domestic bonds $B_{H,-1}$ and foreign bonds $B_{F,-1}$ at $t = -1$, and obtains an endowment $y_t$, which is in the form of a consumption good, for every period $t = 0, 1, \cdots$ . Consuming the real commodity $c_t$ and holding real money $m_t$ $(= M_t/p_t)$ yield utility for the consumer during the period. The preference relationship over them is represented by a period-wise utility function, which is thought of in a logarithmic form. The consumer discounts his future utilities with a discount factor $\beta \in (0, 1)$ , and maximizes his life-time utility. (See (6).) The consumer must be subject to two types of constraints: The first is that of flow budget constraints

$$\begin{aligned} & M_t + B_{Ht} + e_t B_{Ft} \\ \leq \; & p_t(y_t - \tau_t - c_t) + M_{t-1} + (1 + i_{t-1})B_{Ht-1} + e_t(1 + i^*_{t-1})B_{Ft-1}, \end{aligned} \tag{3}$$

$t = 0, 1, \cdots$ . The second one is the NPG condition

$$\liminf_{T\to\infty} \left( \prod_{j=1}^{T} \frac{1}{1 + r_{j-1}} \right) \frac{A_T}{p_T} \geq 0, \tag{4}$$

where

$$A_t = M_{t-1} + (1 + i_{t-1})\, B_{Ht-1} + e_t(1 + i^*_{t-1})B_{Ft-1} \tag{5}$$

is the consumer's financial asset at the beginning of the period $t$ and where $1 + r_t = (1 + i_t)\, p_t/p_{t+1}$ is the real interest rate from period $t$ to $t + 1$. The flow budget constraints and the NPG condition are integrated into an intertemporal budget constraint. (See (6).)

The consumer's behavior is summarized by the following maximizing problem:

$$\max_{\{c_t,\, M_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\log c_t + \gamma \log m_t\right) \tag{6}$$

$$\text{s.t.} \ \sum_{t=0}^{\infty} \left( \prod_{j=1}^{t} \frac{1}{1+r_{j-1}} \right) (c_t + \delta_t m_t) \leq w_0,$$

$$\text{where } w_0 = \frac{A_0}{p_0} + \sum_{t=0}^{\infty} \left( \prod_{j=0}^{t} \frac{1}{1+r_{j-1}} \right) (y_t - \tau_t),$$

where $\gamma > 0$ and where $\delta_t = i_t / (1 + i_t)$ is the opportunity cost of holding real money at period $t$.

Similarly, the consumer in country $F$ resolves an inter-temporal maximizing problem. The flow budget constraints of the foreign consumer are

$$\begin{aligned} & M_t^* + B_{Ft}^* + (1/e_t)\, B_{Ht}^* \\ \leq \ & p_t^*(y_t^* - \tau_t^* - c_t^*) + M_{t-1}^* + (1 + i_{t-1}^*) B_{Ft-1}^* + (1/e_t)\,(1 + i_{t-1}) B_{Ht-1}^*, \end{aligned} \tag{7}$$

$t = 0, 1, 2, \cdots$ . Note that $B_{Ht}^*$ and $B_{Ft}^*$ are the bond held by the consumer in country $F$, which are issued by the governments of countries $H$ and $F$, respectively. The NPG condition is

$$\lim_{T \to \infty} \inf \left( \prod_{j=1}^{T} \frac{1}{1+r_{j-1}^*} \right) \frac{A_T^*}{p_T^*} \geq 0, \tag{8}$$

where

$$A_t^* = M_{t-1}^* + \left(1 + i_{t-1}^*\right) B_{Ft-1}^* + (1/e_t)\,(1 + i_{t-1}) B_{Ht-1}^* \tag{9}$$

and where $1 + r_t^* = (1 + i_t^*)\, p_t^* / p_{t+1}^*$. The maximizing problem for the foreign resident is

$$\max_{\{c_t^*,\, M_t^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\log c_t^* + \gamma^* \log m_t^*\right) \tag{10}$$

$$\text{s.t.} \ \sum_{t=0}^{\infty} \left( \prod_{j=1}^{t} \frac{1}{1+r_{j-1}^*} \right) (c_t^* + \delta_t^* m_t^*) \leq w_0^*,$$

$$\text{where } w_0^* = \frac{A_0^*}{p_0^*} + \sum_{t=0}^{\infty} \left( \prod_{j=0}^{t} \frac{1}{1+r_{j-1}^*} \right) (y_t^* - \tau_t^*),$$

where $\gamma^* > 0$ and $\delta_t^* = i_t^* / (1 + i_t^*)$. The discount factor of the future utilities $\beta \in (0, 1)$ is common to consumers in both countries $H$ and $F$.

The government in country $H$ sets up a stream of policy variables $\{M_t^g, B_t^g, g_t, \tau_t\}$. The stream must subject to flow budget constraints

$$M_t^g + B_t^g = p_t (g_t - \tau_t) + M_{t-1}^g + (1 + i_{t-1}) B_{t-1}^g, \quad t = 0, 1, \cdots . \tag{11}$$

In much the same way, the foreign government is subject to flow budget constraints

$$M_t^{g*} + B_t^{g*} = p_t^* (g_t^* - \tau_t^*) + M_{t-1}^{g*} + \left(1 + i_{t-1}^*\right) B_{t-1}^{g*}, \quad t = 0, 1, \cdots . \tag{12}$$

The stream of price and allocation of the economy is determined so that all markets are cleared simultaneously. Market clearing conditions are described as

$$\begin{aligned} c_t + c_t^* + g_t + g_t^* &= y_t + y_t^*; \\ M_t = M_t^g; &\qquad M_t^* = M_t^{g*}; \\ B_{Ht} + B_{Ht}^* + e_t B_{Ft} + e_t B_{Ft}^* &= B_t^g + e_t B_t^{g*}; \end{aligned} \tag{13}$$

for every $t = 0, 1, \cdots$ .

# 3 Equilibrium Path

This section explicitly presents an equilibrium path without showing a detailed derivation process. For the formal proof, please refer to Kondo [2007] or Neumeyer and Yano [1995]. Although Kondo [2007] deals with a closed economy version of the present model, it is easily ascertained that the equilibrium path of this model can be derived in almost the same way as Kondo [2007]. *The proof can also be found in Appendix I in the present paper.*

To obtain a closed-form solution, the following assumptions are required.

- **Assumptions.**

  **A.** $y_t = y, \ g_t = g, \ \tau_t = \tau, \ y_t^* = y^*, \ g_t^* = g^*, \ \tau_t^* = \tau^*$ for every $t = 0, 1, \cdots$ .

  **B.** $y > \tau \geq 0, \ y > g \geq 0, \ y^* > \tau^* \geq 0, \ y^* > g^* \geq 0.$

  **C.** $M_t^g / M_{t-1}^g = 1 + \mu$ and $M_t^{g*} / M_{t-1}^{g*} = 1 + \mu^*$ for every $t = 0, 1, \cdots$ .

  **D.** $\beta < 1 + \mu$ and $\beta < 1 + \mu^*$.

  **E.** $\lim_{T \to \infty} \left( \prod_{j=1}^{T} \frac{1}{1 + i_{j-1}} \right) = 0$ and $\lim_{T \to \infty} \left( \prod_{j=1}^{T} \frac{1}{1 + i_{j-1}^*} \right) = 0.$

Under these assumptions, the next lemma is established.

**Lemma 1** *Given the initial prices $p_0$ and $p_0^*$, an equilibrium path can be expressed as below*

$$(i)\ 1+r_t = 1+r_t^* = \frac{1}{\beta}; \qquad (ii)\ 1+i_t = \frac{1+\mu}{\beta}; \qquad 1+i_t^* = \frac{1+\mu^*}{\beta};$$

$$(iii)\ p_t = (1+\mu)^t p_0; \qquad p_t^* = (1+\mu^*)^t p_0^*; \qquad (iv)\ e_t = \left(\frac{1+\mu}{1+\mu^*}\right)^t \frac{p_0}{p_0^*};$$

$$(v)\ c_t = \frac{1-\beta}{1+\gamma}\frac{A_0}{p_0} + \frac{y-\tau}{1+\gamma}; \qquad c_t^* = \frac{1-\beta}{1+\gamma^*}\frac{A_0^*}{p_0^*} + \frac{y^*-\tau^*}{1+\gamma^*};$$

$$(vi)\ m_t = \frac{\gamma}{1+\gamma}\frac{1+\mu}{1+\mu-\beta}\left[(1-\beta)\frac{A_0}{p_0} + y - \tau\right];$$

$$m_t^* = \frac{\gamma^*}{1+\gamma^*}\frac{1+\mu^*}{1+\mu^*-\beta}\left[(1-\beta)\frac{A_0^*}{p_0^*} + y^* - \tau^*\right];$$

$$(vii)\ M_t = (1+\mu)^{t+1} M_{-1}^g; \qquad M_t^* = (1+\mu^*)^{t+1} M_{-1}^{g*};$$

$$(viii)\ B_t^g = \left(\frac{1+\mu}{\beta}\right)^t (1+i_{-1})\, B_{-1}^g + (1+\mu)^t \left[p_0(g-\tau) - \mu M_{-1}^g\right] \sum_{s=0}^{t} \beta^{-s};$$

$$B_t^{g*} = \left(\frac{1+\mu^*}{\beta}\right)^t \left(1+i_{-1}^*\right) B_{-1}^{g*} + (1+\mu^*)^t \left[p_0^*(g^*-\tau^*) - \mu^* M_{-1}^{g*}\right] \sum_{s=0}^{t} \beta^{-s};$$

*for every* $t = 0, 1, \cdots$.

Three remarks about Lemma 1 are given. First, this study traces the impact of an unexpectedly implemented monetary policy change at time $t=-1$ over the equilibrium path. Market reaction to the policy change is speedily reflected in prices $p_0, p_0^*, e_0, i_0, i_0^*, \cdots$ and in the initial assets of the consumers

$$\begin{aligned} A_0 &= M_{-1} + (1+i_{-1})\, B_{H,-1} + e_0(1+i_{-1}^*) B_{F,-1} \text{ and} \\ A_0^* &= M_{-1}^* + \left(1+i_{-1}^*\right) B_{F,-1}^* + (1/e_0)\,(1+i_{-1}) B_{H,-1}^*. \end{aligned}$$

The initial assets are influenced through the change of $e_0$, while $M_{-1}$, $i_{-1}$, $B_{H,-1}$, $i_{-1}^*$, $B_{F,-1}$ etc are exogenously given for our analyses.

Second, there exists a relationship between important variables like the "Trinity". As demonstrated in (*ii*) and (*iii*) of the lemma, a high level of the money supply growth rate, an acceleration of inflation and a high nominal interest rate are equivalent with each other in the equilibrium.

Third, it is clear from Lemma 1 that if $p_0 > 0$, $p_0^* > 0$ and $\beta$ is sufficiently close to 1, all endogenous variables except for $B_t^g$ and $B_t^{g*}$ become positive. Assuming that consumers discount their future utilities sufficiently weakly, we explore, in the next section, under what conditions $p_0 > 0$ and $p_0^* > 0$ are guaranteed.

The initial prices $p_0$ and $p_0^*$ are also explicitly derived. By $(vi)$ in Lemma 1, we have

$$\begin{aligned} p_0 &= \frac{(1+\gamma)(1+\mu-\beta) M_{-1}^g - \gamma (1-\beta) A_0}{\gamma(y-\tau)}; \\ p_0^* &= \frac{(1+\gamma^*)(1+\mu^*-\beta) M_{-1}^{g*} - \gamma^* (1-\beta) A_0^*}{\gamma^*(y^*-\tau^*)}. \end{aligned} \tag{14}$$

The equations in (14), together with (1)(5)(9), are transformed as

$$\begin{aligned} p_0 &= \frac{1}{\gamma} \frac{(1+\mu-\beta+\gamma\mu) M_{-1}^g - \gamma (1-\beta)(1+i_{-1}) B_{H,-1}}{(y-\tau) + (1-\beta)(1/p_0^*)\left(1+i_{-1}^*\right) B_{F,-1}}; \\ p_0^* &= \frac{1}{\gamma^*} \frac{(1+\mu^*-\beta+\gamma^*\mu^*) M_{-1}^{g*} - \gamma^* (1-\beta)\left(1+i_{-1}^*\right) B_{F,-1}^*}{(y^*-\tau^*) + (1-\beta)(1/p_0)(1+i_{-1}) B_{H,-1}^*}. \end{aligned}$$

For the sake of simplicity, we introduce new parameters. Define $\alpha$ and $\alpha^* \, (\in [0,1])$ by

$$\begin{aligned} B_{H,-1} &\equiv \alpha B_{-1}^g, \qquad B_{H,-1}^* \equiv (1-\alpha) B_{-1}^g, \\ B_{F,-1}^* &\equiv \alpha^* B_{-1}^{g*}, \qquad B_{F,-1} \equiv (1-\alpha^*) B_{-1}^{g*}, \end{aligned} \tag{15}$$

that represent levels of inter-connection between the two countries at the initial time point $t = -1$. Note that $\alpha$ and $\alpha^*$ are exogenously given for our analyses, as are other initial conditions. Further, we define

$$\begin{aligned} \mathcal{M} &\equiv (1+\mu-\beta+\gamma\mu) M_{-1}^g, \qquad \mathcal{B} \equiv \gamma (1-\beta)(1+i_{-1}) B_{-1}^g, \\ \mathcal{M}^* &\equiv (1+\mu^*-\beta+\gamma^*\mu^*) M_{-1}^{g*}, \qquad \mathcal{B}^* \equiv \gamma^* (1-\beta)\left(1+i_{-1}^*\right) B_{-1}^{g*}. \end{aligned} \tag{16}$$

With the new notations, the initial prices can be written down relatively simply:

$$\begin{aligned} p_0 &= \frac{1}{\gamma} \frac{(\mathcal{M}-\mathcal{B}\alpha)(\mathcal{M}^*-\mathcal{B}^*\alpha^*) - \mathcal{B}(1-\alpha)\mathcal{B}^*(1-\alpha^*)}{(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*) + (y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)}, \\ p_0^* &= \frac{1}{\gamma^*} \frac{(\mathcal{M}-\mathcal{B}\alpha)(\mathcal{M}^*-\mathcal{B}^*\alpha^*) - \mathcal{B}(1-\alpha)\mathcal{B}^*(1-\alpha^*)}{(y-\tau)\mathcal{B}(1-\alpha) + (y^*-\tau^*)(\mathcal{M}-\mathcal{B}\alpha)}. \end{aligned} \tag{17}$$

Lemma 1 and (17) show the equilibrium path.

# 4 NPG Condition

This section offers a set of parameter conditions required for our analyses to be meaningful. First, all endogenous variables need to be positive. Further, the NPG condition must be compatible with the equilibrium demonstrated in Lemma 1 and (17). The NPG condition plays a crucial role in this paper; thus particular attention is placed on it in the next section.

At the outset, we make some additional assumptions.

- **Assumptions**

  **D′.** $(\gamma + \beta) / (1 + \gamma) < 1 + \mu, \ \ (\gamma^* + \beta) / (1 + \gamma^*) < 1 + \mu^*.$

  **F.** $\mathcal{M} - \mathcal{B} > 0, \ \ \mathcal{M}^* - \mathcal{B}^* > 0$, where $(\mathcal{M}, \mathcal{M}^*, \mathcal{B}, \mathcal{B}^*)$ are defined in (16).

Assumption $D'$ strengthens Assumption $D$ to guarantee $\mathcal{M} > 0$ and $\mathcal{M}^* > 0$. Assumption $F$ may be interpreted that money has already been sufficiently supplied at the initial time point in both countries.

Non-negativity conditions, the conditions for all endogenous variables become positive, are focused. The expressions (17) show that if both $\alpha$ and $\alpha^*$ are close enough to 1, the conditions $p_0 > 0$ and $p_0^* > 0$ are simultaneously held under Assumption $F$. This, together with $\beta$ sufficiently close to 1, guarantees that all endogenous variables are positive.

The NPG condition for the government is that it is necessary for its financial liability to be repayable over an infinite time horizon. The NPG condition for the domestic government is identified as

$$\limsup_{T\to\infty} \left( \prod_{t=1}^{T} \frac{1}{1 + r_{t-1}} \right) \frac{D_T^g}{p_T} \leq 0, \tag{18}$$

where $D_T^g = M_{T-1}^g + (1 + i_{T-1}) B_{T-1}^g$ represents the government's financial liabilities. (For this point, see Hamilton and Flavin [1986].) Notice that the path of variables in (18) $\{r_t, i_t, p_t, M_t^g, B_t^g\}$ is along an equilibrium path.

We rewrite (18) using Lemma 1 and (17). Before that, note that if both $\alpha$ and $\alpha^*$ are close enough to 1, it holds that

$$\begin{gathered}(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)>0 \text{ and} \\ (y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*) \\ +(\tau-g)\{\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\}>0.\end{gathered} \tag{19}$$

With the condition $p_0 > 0$ and (19), we can derive the NPG condition in much the same way as in an appendix in Kondo [2007]:

$$\begin{aligned} NPG &\Longleftrightarrow \mathcal{B} \leq \gamma p_0(\tau-g)+\theta\mathcal{M}. \\ &\Longleftrightarrow \mathcal{B} \leq \mathcal{M}\frac{\begin{bmatrix}(\tau-g)(\mathcal{M}^*-\mathcal{B}^*\alpha^*) \\ +(y-\tau)\theta(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\theta\mathcal{B}^*(1-\alpha^*)\end{bmatrix}}{\begin{bmatrix}(\tau-g)\{\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\} \\ +(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\end{bmatrix}},\end{aligned} \tag{20}$$

where

$$\theta \equiv \frac{\gamma\mu}{1+\mu-\beta+\gamma\mu}. \tag{21}$$

*The detailed derivation process is provided in Appendix II.* The following lemma has been established.

**Lemma 2** *The NPG condition for the domestic government in a DGE is given by*

$$\begin{aligned} B^g_{-1} \leq & \frac{1}{\gamma(1-\beta)(1+i_{-1})}\mathcal{M}\cdot \\ & \frac{\begin{bmatrix}(\tau-g)(\mathcal{M}^*-\mathcal{B}^*\alpha^*) \\ +(y-\tau)\theta(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\theta\mathcal{B}^*(1-\alpha^*)\end{bmatrix}}{\begin{bmatrix}(\tau-g)\{\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\} \\ +(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\end{bmatrix}} \equiv \varphi,\end{aligned} \tag{22}$$

*where* $(\alpha,\alpha^*)$, $(\mathcal{M},\mathcal{M}^*,\mathcal{B}^*)$ *and* $\theta$ *are defined in (15), (16) and (21), respectively.*

The next section analyzes how unexpected monetary policy changes conducted at the beginning of the period 0 affect the market outcome and the upper bound of the national debt $\varphi$. More concretely, signs of $\partial\varphi/\partial\mu$ and $\partial\varphi/\partial\mu^*$ are investigated.

We check that

$$\gamma(1-\beta)(1+i_{-1})\varphi \ \big(= \text{an upper bound for } \mathcal{B} \text{ (not for } B^g_{-1})\big) < \mathcal{M}. \quad (23)$$

Otherwise, or if $(\mathcal{B}<)\mathcal{M} < \gamma(1-\beta)(1+i_{-1})\varphi$, the validity of the analysis exploring the possible range of $B^g_{-1}$ that is compatible with the NPG condition is not necessarily guaranteed because we exogenously assume $\mathcal{B} < \mathcal{M}$ (Assumption $F$). Since $\theta < 1$, if $\alpha$ and $\alpha^*$ are close to 1, (23) is met.[1]

In what follows, $\alpha$ and $\alpha^*$ are both assumed to be close enough to 1 to analyze NPG conditions, (20) without being concerned about the non-negativity condition and (23).

**Case of** $\alpha^* = 1$

Here, a special case in which the consumer in the home country initially has made his portfolio without the foreign country's bond is focused on. In this case, the foreign country's monetary policy has no effect on the upper bound $\varphi$. Indeed, substituting $\alpha^* = 1$ into (22) yields

$$B^g_{-1} \leq \frac{1}{\gamma(1-\beta)(1+i_{-1})}\mathcal{M}\frac{(\tau-g)+(y-\tau)\theta}{(\tau-g)\alpha+(y-\tau)}, \quad (24)$$

which shows that the upper bound $\varphi$ is not affected by the foreign country's monetary policy $\mu^*$. Thus, an inflation spiral through the channel this study focuses on does not occur. In what follows, $\alpha, \alpha^* \neq 1$ is assumed to investigate a policy interaction through cross-border bond holding in the markets.

[1] Since $\mathcal{M} > 0$, it is sufficient to check that

$$\begin{aligned} & (\tau-g)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y-\tau)\theta(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\theta\mathcal{B}^*(1-\alpha^*) \\ < \ & (\tau-g)\{\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\} \\ & +(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*). \end{aligned}$$

Equivalently,

$$\begin{aligned} & (\tau-g)(1-\alpha)\{(\mathcal{M}^*-\mathcal{B}^*\alpha^*)-\mathcal{B}^*(1-\alpha^*)\} \\ < \ & (y-\tau)(1-\theta)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)(1-\theta)\mathcal{B}^*(1-\alpha^*). \end{aligned}$$

# 5 International Linkage of Inflation Rates

This section sets out the main analyses presented in this paper. While the result of the previous section shows how the upper bound of financial liabilities $\varphi$ depends on the monetary policies of both countries, this section demonstrates that there exists a certain range of parameter values for which the monetary policies of both countries positively correlate. More precisely, if a government's budget deficit is not extremely large, the conditions $\partial\varphi/\partial\mu^* < 0$ and $\partial\varphi/\partial\mu > 0$ hold simultaneously. The former condition $\partial\varphi/\partial\mu^* < 0$ implies that the foreign government's expanding monetary policy impinges on the domestic government's financial standing in the sense that it reduces the upper bound $\varphi$. This strongly motivates the domestic government to print more money if $\partial\varphi/\partial\mu > 0$. That is, expanding monetary policy by the domestic government is induced by a similar policy of the foreign government. Once the possibility in which policies correlate positively is demonstrated, the examination of the case of negative interaction is straightforward.

## 5.1 Condition for $\partial\varphi/\partial\mu > 0$

Calculation from (22) yields

$$\frac{\partial\varphi}{\partial\mu} \gtreqless 0 \Longleftrightarrow g-\tau \lesseqgtr \frac{(y-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+\left(y^*-\tau^*\right)\mathcal{B}^*(1-\alpha^*)}{\mathcal{M}^*-\mathcal{B}^*\alpha^*}\cdot \left[\theta+\frac{\gamma}{1+\gamma}\frac{1-\beta}{1+\mu-\beta+\gamma\mu}\right] \equiv \Psi. \quad (25)$$

*The detailed derivation process is provided in Appendix III.* The right hand side of (25), $\Psi$, is a critical level for the fiscal deficit $g-\tau$ under which the domestic government's expanding monetary policy improves its financial standing in the sense that it raises the upper bound $\varphi$.

Conversely, if $\Psi < g-\tau$, printing lots of money makes the fiscal standing worse. Although that seems intuitively implausible, fiscal policy with an extremely high level of fiscal deficit yields $\partial\varphi/\partial\mu < 0$ along an equilibrium path.[2] An intuitive explanation for this point is as follows. As pointed out right after Lemma 1, a lower value for the money supply growth rate directly

[2] This point has already pointed out by Kondo and Kitaura [2008] using a closed economy version of the present model.

implies both deflation (or slowdown of inflation) and a low nominal interest rate. Under the condition $\Psi < g - \tau$, a low interest payment obligation is more attractive for the government than printing money to raise the upper bound $\varphi$.

We sum the result (25) as a lemma.

**Lemma 3** *The government's expanding monetary policy improves its financial standing if and only if*

$$g - \tau < \frac{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)}{\mathcal{M}^* - \mathcal{B}^*\alpha^*} \cdot \left[\theta + \frac{\gamma}{1+\gamma}\frac{1-\beta}{1+\mu-\beta+\gamma\mu}\right] \equiv \Psi, \quad (26)$$

*where* $\alpha^*$ *and* $(\mathcal{M}^*, \mathcal{B}^*)$ *are defined in (15) and (16), respectively.*

To compare this characterization of $\partial\varphi/\partial\mu > 0$ with the condition for $\partial\varphi/\partial\mu^* < 0$, take a limit:

$$\Psi \to \frac{\gamma}{1+\gamma}(y-\tau) \quad \text{as } \alpha^* \to 1 \text{ and } \beta \to 1. \qquad (27)$$

The equation (27) is referred to in the next subsection.

## 5.2 Condition for $\partial\varphi/\partial\mu^* < 0$

Let's turn to the characterization of the condition $\partial\varphi/\partial\mu^* < 0$. A long and somewhat tedious calculation from (22) yields

$$\frac{\partial\varphi}{\partial\mu^*} \lesseqqgtr 0 \Longleftrightarrow (g-\tau)B^{g*}_{-1}\left[\begin{array}{c}(1-\alpha)\theta(y-\tau) + (1-\alpha\theta)(y^*-\tau^*) \\ -(1-\alpha)(g-\tau)\end{array}\right] \gtreqqless 0. \qquad (28)$$

*The detailed derivation process is provided in Appendix IV.*

For simplicity, assume that the foreign government is a debtor:

- **Assumption G.** $B^{g*}_{-1} > 0$.

It is straightforward from (28) and Assumption $G$ that the following lemma holds.

**Lemma 4** *The foreign country's expanding monetary policy has a negative impact on the critical level of the domestic government's fiscal deficit that is compatible with the NPG condition and DGE, i.e.,* $\partial\varphi/\partial\mu^* < 0$, *if and only if*

$$0 < g - \tau < \theta (y - \tau) + \frac{1 - \alpha\theta}{1 - \alpha} (y^* - \tau^*) \equiv \Phi. \tag{29}$$

*Moreover,* $g - \tau < 0$ *implies* $\partial\varphi/\partial\mu^* > 0$, *where* $\alpha$ *and* $\theta$ *are defined in (15) and (21), respectively.*

Lemmas 3 and 4 directly show that $\partial\varphi/\partial\mu$ and $\partial\varphi/\partial\mu^*$ change their sign when $g - \tau$ crosses over $\Psi$ and $\Phi$, respectively. To summarize the analyses, it is necessary to investigate the behavior of $\Psi$ and $\Phi$ within the limit we are focusing on:

$$\Phi \to \frac{\gamma}{1 + \gamma} (y - \tau) + \frac{1}{1 + \gamma} (y^* - \tau^*) \quad \text{as } \alpha \to 1 \text{ and } \beta \to 1. \tag{30}$$

Comparing (27) and (30), we find that $\Psi < \Phi$ in the near limit. We summarize this result as a lemma.

**Lemma 5** *If* $\alpha$ *and* $\alpha^*$ *are close enough to 1, and if the discount factor* $\beta$ *is close enough to 1, it holds that* $0 < \Psi < \Phi$, *where* $(\alpha, \alpha^*)$, $\Psi$ *and* $\Phi$ *are defined in (15) (26) and (29), respectively.*

## 5.3 Main Results

Our findings, Lemma 3 and 4 with Lemma 5, can be visualized as follows:

| | | $\partial\varphi/\partial\mu$ | $\partial\varphi/\partial\mu^*$ |
|---|---|---|---|
| *(I)* | $\Phi < g - \tau$ | $-$ | $+$ |
| *(II)* | $\Psi < g - \tau < \Phi$ | $-$ | $-$ |
| *(III)* | $0 < g - \tau < \Psi$ | $+$ | $-$ |
| *(IV)* | $g - \tau < 0$ | $+$ | $+$ |

(31)

where $\Psi$ and $\Phi$ are defined in (26) and (29), respectively. There are various patterns in which foreign and domestic policies (and inflation rates) closely correlate. According to the classification shown in (31), we explain this point in order.

See $(III)$ at the outset. The foreign country's inflation caused by an expanded monetary policy is easily transmitted to the home country. Consider the situation in which the foreign government unexpectedly increases its money supply at the beginning of the period 0 to compensate for its fiscal deficit. Then, the upper bound of the domestic government $\varphi$ is damaged since $\partial\varphi/\partial\mu^* < 0$, motivating the domestic government to also print more money since $\partial\varphi/\partial\mu > 0$. In other words, incentives to cover expenditure or to raise the upper bound $\varphi$ result in printing extra money in both countries and, equivalently, in synchronized inflation at the world level.

In the case of $(II)$, foreign acceleration of inflation results in domestic deflation (or slowdown of inflation). Although the foreign country's expanding monetary policy impinges on the domestic government's financial standing ($\partial\varphi/\partial\mu^* < 0$), the domestic government is not willing to print more money to finance its fiscal deficit since $\partial\varphi/\partial\mu < 0$. Rather, foreign inflation motivates the domestic government to reduce money supply for the purpose of controlling nominal interest payment obligation to a low level. (As already pointed out, a lower value of $\mu$, deflation and a low nominal interest rate are equivalent with each other in the equilibrium.) Thus, if the government deficit is not low, deflation can come from foreign inflation.

$(I)$. International synchronized deflation may emerge. With a high level of fiscal deficit, more precisely $\Psi^* < g^* - \tau^*$, the foreign government has an incentive to reduce money because that improves its fiscal standing, where $\Psi^*$ is defined in an analogous manner as (26). Further, if the domestic government's fiscal deficit is extremely high, i.e. $\Phi < g - \tau$, foreign deflation impinges on the fiscal standing for the domestic government, motivating it to reduce the money supply since $\partial\varphi/\partial\mu < 0$. In other words, under high levels of fiscal deficits in both countries, deflation harmonizes internationally.

In the case of $(IV)$, the domestic economy can defend itself against monetary disturbances from abroad. The domestic government has no incentive to expand its monetary policy as a countermeasure against foreign inflation because the foreign government's inflation-oriented policy rather improves the domestic government's financial standing. Also, a foreign government has no incentive to reduce money, as long as its budget deficit is not so high $g^* - \tau^* < \Psi^*$. To conclude, fiscal policy stabilizes the world's monetary system in such scenarios.

The main results of this paper can be summarized as the following theorem.

**Theorem 1** *Assume that both* $\alpha, \alpha^* \in [0, 1)$ *are close enough to 1, and that a foreign government's fiscal deficit is not so huge, more precisely* $g^* - \tau^* < \Psi^*$*, where the parameters* $(\alpha, \alpha^*)$ *are defined in (15) and* $\Psi^*$ *is defined in an analogous manner as (26). Then, the following hold. (The parameters* $\Psi$ *and* $\Phi$ *are defined in (26) and (29), respectively. )*

*If* $0 < g - \tau < \Psi$*, foreign inflation transmits to a domestic economy beyond the national border.*

*If* $\Psi < g - \tau < \Phi$*, deflation occurs resulting from foreign inflation.*

*If* $\Phi < g - \tau$*, foreign deflation transmits to the domestic economy, although the foreign government has no incentive to cause deflation.*

*If* $g - \tau < 0$*, the monetary system tied to the world economy is stable in the sense that inflation pressure from abroad does not result in domestic inflation, while a foreign government has no incentive to cause deflation.*

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# 6 Appendix

In the Appendix, we demonstrate Lemma 1-4, which are too long to allocate in the body part of this paper. As mentioned in the introduction, the Appendix is prepared not for publication but for referee-process only.

To begin with, we reconsider the consumers's maximizing problem. The home country's consumer sequentially maximizes his lifetime utilities at each period:

$$
\begin{aligned}
&\max_{\{c_{t+s},\, M_{t+s}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^{s} \left(\log c_{t+s} + \gamma \log m_{t+s}\right) \qquad (32)\\
&\text{s.t. } \sum_{s=0}^{\infty} \left( \prod_{j=1}^{s} \frac{1}{1 + r_{t+j-1}} \right) \left(c_{t+s} + \delta_{t+s} m_{t+s}\right) \leq w_t,\\
&\text{where } w_t = \frac{A_t}{p_t} + \sum_{s=0}^{\infty} \left( \prod_{j=0}^{s-1} \frac{1}{1 + r_j} \right) \left(y_{t+s} - \tau_{t+s}\right),
\end{aligned}
$$

$t = 0, 1, \cdots$ . For the problem (6) to be solved, it is necessary that the consumer maximizes (32) at each periods. We define an operator taking the discounted value $\widetilde{x}_t = \sum_{s=0}^{\infty} \left( \prod_{j=1}^{s} \frac{1}{1+r_{t+j-1}} \right) x_{t+s}$ for given $t$ and $\{x_{t+s}\}_{s=0}^{\infty}$. Using this operator, we can rewrite $w_t = A_t/p_t + \widetilde{y}_t - \widetilde{\tau}_t$.

The maximizing problem and the wealth constraint for the consumer in the country $F$ can be described in just an analogously way.

## 6.1 Appendix I — Proof of Lemma 1

First, we derive the optimal consumption path (36). To this end, set up the Lagrange function associating the maximizing problem (32) as

$$
\begin{aligned}
L_t &= \sum_{s=0}^{\infty} \beta^{s} \left(\log c_{t+s} + \gamma \log m_{t+s}\right)\\
&\quad + \lambda_t \left[ w_t - \sum_{s=0}^{\infty} \left( \prod_{j=1}^{s} \frac{1}{1 + r_{t+j-1}} \right) \left(c_{t+s} + \delta_{t+s} m_{t+s}\right) \right],
\end{aligned}
$$

for $t = 0, 1, \cdots$ where $\lambda_t$ is the Lagrange multiplier. First order conditions can be obtain as

$$\frac{\partial L_t}{\partial c_{t+s}} = 0: \quad c_{t+s} = \beta^s \left( \prod_{j=1}^{s} (1 + r_{t+j-1}) \right) \frac{1}{\lambda_t}, \tag{33}$$

$$\frac{\partial L_t}{\partial M_{t+s}} = 0: \quad \delta_{t+s} m_{t+s} = \beta^s \left( \prod_{j=1}^{s} (1 + r_{t+j-1}) \right) \frac{\gamma}{\lambda_t}, \tag{34}$$

$$\frac{\partial L_t}{\partial \lambda_t} = 0: \quad w_t = \sum_{\tau=0}^{\infty} \left( \prod_{j=1}^{\tau} \frac{1}{1 + r_{t+j-1}} \right) (c_{t+\tau} + \delta_{t+\tau} m_{t+\tau}), \tag{35}$$

for $s, t = 0, 1, \cdots$ . Letting $s = 0$ in (33) and (34), we have

$$c_t = \frac{1}{\lambda_t}, \qquad \delta_t m_t = \frac{\gamma}{\lambda_t},$$

for $t = 0, 1, \cdots$ . Substituting into (35) yields

$$\lambda_t = \frac{1 + \gamma}{1 - \beta} \frac{1}{w_t}$$

for $t = 0, 1, \cdots$ . Thus, we obtain the optimal consumption path:

$$\begin{aligned} c_t &= \frac{1}{1+\gamma} (1 - \beta) w_t; \qquad \delta_t m_t = \frac{\gamma}{1+\gamma} (1 - \beta) w_t; \\ c_t^* &= \frac{1}{1+\gamma^*} (1 - \beta) w_t^*; \qquad \delta_t^* m_t^* = \frac{\gamma^*}{1+\gamma^*} (1 - \beta) w_t^*; \end{aligned} \tag{36}$$

for $t = 0, 1, \cdots$ .

As was seen in (36), to get the consumption path we need to obtain the dynamics of the real wealth $\{w_t\}$. In the equilibrium, the dynamics of the consumer's real wealth are determined as following:

**Sublemma 1** *It holds that* $w_{t+1} = \beta (1 + r_t) w_t$ *and* $w_{t+1}^* = \beta (1 + r_t^*) w_t^*$ *for every* $t = 0, 1, \cdots$ .

**Proof.** By the flow budget constraint, it holds that

$$c_t + \delta_t m_t = y_t - \tau_t + \frac{A_t}{p_t} - \frac{1}{1 + r_t} \frac{A_{t+1}}{p_{t+1}}.$$

Since (36) holds, the LHS satisfies

$$LHS = (1-\beta)\, w_t.$$

Moreover, since $(\widetilde{y}_t - \widetilde{\tau}_t) - \frac{1}{1+r_t}(\widetilde{y}_{t+1} - \widetilde{\tau}_{t+1}) = y_t - \tau_t$, by the definition of the wealth constraint, we obtain

$$\begin{aligned} RHS &= y_t - \tau_t + w_t - (\widetilde{y}_t - \widetilde{\tau}_t) - \frac{1}{1+r_t}\left[w_{t+1} - (\widetilde{y}_{t+1} - \widetilde{\tau}_{t+1})\right] \\ &= w_t - \frac{1}{1+r_t} w_{t+1}. \end{aligned}$$

Hence, $w_{t+1} = \beta\,(1+r_t)\, w_t$. ■

The real expenditure to the consumption and the money balance along the equilibrium path are determined by (36) and Sublemma 1.

**Sublemma 2** *It holds that*

$$\begin{aligned} c_{t+1} &= \beta\,(1+r_t)\, c_t, \qquad \delta_{t+1} m_{t+1} = \beta\,(1+r_t)\,\delta_t m_t \\ c^*_{t+1} &= \beta\,(1+r_t)\, c^*_t, \qquad \delta^*_{t+1} m^*_{t+1} = \beta\,(1+r^*_t)\,\delta^*_t m^*_t \end{aligned}$$

*for every* $t = 0, 1, \cdots$.

**Proof.** By (36) and Sublemma 1, the following holds

$$c_{t+1} = \frac{1-\beta}{1+\gamma}\beta\,(1+r_t)\, w_t = \frac{1-\beta}{1+\gamma}\beta\,(1+r_t)\,\frac{1+\gamma}{1-\beta} c_t = \beta\,(1+r_t)\, c_t.$$

In much the same way, we can obtain $\delta_{t+1} m_{t+1} = \beta\,(1+r_t)\,\delta_t m_t$. ■

The real interest rate, which is crucial to the equilibrium path as well as the real wealth, is essentially influenced by the discount factor $\beta$ and the disposable products in the present model.

**Sublemma 3** *It holds that*

$$1 + r_t = 1 + r^*_t = \frac{1}{\beta}\frac{y_{t+1} + y^*_{t+1} - g_{t+1} - g^*_{t+1}}{y_t + y^*_t - g_t - g^*_t}$$

*for every* $t = 0, 1, \cdots$.

**Proof.** By the PPP (1) and the interest rate parity (2), it holds that $r_t = r_t^*$ for every $t = 0, 1, 2, \cdots$ .

Further, by the market clearing condition (13) and Sublemma 2, it holds that

$$\frac{y_{t+1} + y_{t+1}^* - g_{t+1} - g_{t+1}^*}{y_t + y_t^* - g_t - g_t^*} = \frac{c_{t+1} + c_{t+1}^*}{c_t + c_t^*} = \beta (1 + r_t) .$$

This ends the proof. ■

The nominal variables (the price level, the opportunity cost of the real money balance and the nominal interest rate) must satisfy the following relationships (Sublemma 4, 5).

**Sublemma 4** *It holds that*

$$\begin{aligned} \frac{p_{t+1}}{p_t} &= \frac{M_{t+1}^g}{M_t^g} \frac{\delta_{t+1}}{\delta_t} \frac{y_t + y_t^* - g_t - g_t^*}{y_{t+1} + y_{t+1}^* - g_{t+1} - g_{t+1}^*} \\ \frac{p_{t+1}^*}{p_t^*} &= \frac{M_{t+1}^{g*}}{M_t^{g*}} \frac{\delta_{t+1}^*}{\delta_t^*} \frac{y_t + y_t^* - g_t - g_t^*}{y_{t+1} + y_{t+1}^* - g_{t+1} - g_{t+1}^*} \end{aligned}$$

*for every* $t = 0, 1, \cdots$ .

**Proof.** By Sublemma 2 and 3, it holds in an equilibrium that

$$\frac{\delta_{t+1} m_{t+1}}{\delta_t m_t} = \frac{\delta_{t+1}^* m_{t+1}^*}{\delta_t^* m_t^*} = \frac{y_{t+1} + y_{t+1}^* - g_{t+1} - g_{t+1}^*}{y_t + y_t^* - g_t - g_t^*}.$$

Thus, we can obtain the Sublemma. ■

**Sublemma 5** *It holds that*

$$1 + i_t = \frac{1}{\beta} \frac{M_{t+1}^g}{M_t^g} \frac{\delta_{t+1}}{\delta_t}, \qquad 1 + i_t^* = \frac{1}{\beta} \frac{M_{t+1}^{g*}}{M_t^{g*}} \frac{\delta_{t+1}^*}{\delta_t^*}$$

*for every* $t = 0, 1, \cdots$ .

**Proof.** By Sublemma 3 and 4, it holds that

$$\begin{aligned} 1 + i_t &= (1 + r_t) \frac{p_{t+1}}{p_t} \\ &= \frac{1}{\beta} \frac{y_{t+1} + y_{t+1}^* - g_{t+1} - g_{t+1}^*}{y_t + y_t^* - g_t - g_t^*} \frac{\delta_{t+1}}{\delta_t} \frac{M_{t+1}^g}{M_t^g} \frac{y_t + y_t^* - g_t - g_t^*}{y_{t+1} + y_{t+1}^* - g_{t+1} - g_{t+1}^*} \\ &= \frac{1}{\beta} \frac{\delta_{t+1}}{\delta_t} \frac{M_{t+1}^g}{M_t^g}. \end{aligned}$$

■

The next Sublemma is a crucial sub-step for proving Sublemma 7.

**Sublemma 6** *For every* $t = 0, 1, 2, \cdots$ *, the following hold*

$$(i)\ \frac{p_t}{p_{t+1}} = \frac{1}{1+\mu}\frac{m_{t+1}}{m_t}; \qquad \frac{p_t^*}{p_{t+1}^*} = \frac{1}{1+\mu^*}\frac{m_{t+1}^*}{m_t^*};$$

$$(ii)\ \delta_t = 1 - \frac{1}{1+\mu}\frac{1}{1+r_t}\frac{m_{t+1}}{m_t}; \qquad \delta_t^* = 1 - \frac{1}{1+\mu^*}\frac{1}{1+r_t^*}\frac{m_{t+1}^*}{m_t^*};$$

$$(iii)\ \frac{1}{1+\mu}\frac{1}{1+r_t} = \frac{1}{1+i_t}\frac{m_t}{m_{t+1}}; \qquad \frac{1}{1+\mu^*}\frac{1}{1+r_t^*} = \frac{1}{1+i_t^*}\frac{m_t^*}{m_{t+1}^*};$$

**Proof.** $(i)$ Since $\frac{m_{t+1}}{m_t} = \frac{M_{t+1}^g/p_{t+1}}{M_t^g/p_t} = (1+\mu)\frac{p_t}{p_{t+1}}$, the result follows.
$(ii)$ By $(i)$, we can easily demonstrate the desired result:

$$\delta_t = \frac{i_t}{1+i_t} = 1 - \frac{1}{1+i_t} = 1 - \frac{1}{1+r_t}\frac{p_t}{p_{t+1}} = 1 - \frac{1}{1+r_t}\frac{1}{1+\mu}\frac{m_{t+1}}{m_t}.$$

$(iii)$ By the proof of $(ii)$,

$$\frac{1}{1+r_t}\frac{1}{1+\mu}\frac{m_{t+1}}{m_t} = 1 - \delta_t = \frac{1}{1+i_t}.$$

Thus, $(iii)$ holds. ■

Under the assumptions, we obtain the equilibrium level of the real money balance $m_t$ and $m_t^*$.

**Sublemma 7** *It holds that*

$$m_t = \frac{\gamma(1-\beta)}{1+\gamma}w_t\frac{1+\mu}{1+\mu-\beta}, \qquad m_t^* = \frac{\gamma^*(1-\beta)}{1+\gamma^*}w_t^*\frac{1+\mu^*}{1+\mu^*-\beta}$$

*for every* $t = 0, 1, \cdots$ .

**Proof.** Take $t = 0, 1, \cdots$ arbitrary and fix it momentarily. By using (36) and $(ii)$ of Sublemma 6, we can obtain, by induction, that

$$m_t - \left(\prod_{j=1}^{T}\frac{1}{1+\mu}\frac{1}{1+r_{t+j-1}}\right)m_{t+T} = \gamma\sum_{s=0}^{T-1}\left(\prod_{j=1}^{s}\frac{1}{1+\mu}\frac{1}{1+r_{t+j-1}}\right)c_{t+s}$$

for any $T$=1, 2, $\cdots$ . In an equilibrium, the limit of the both side of the above expression as $T \to \infty$ can be obtained as follows. By Assumption C, D and Sublemma 2,

$$\begin{aligned} RHS &= \gamma \sum_{s=0}^{T-1} \left( \prod_{j=1}^{s} \frac{1}{1+\mu} \frac{1}{1+r_{t+j-1}} \right) \beta^s \left( \prod_{j=1}^{s} (1+r_{t+j-1}) \right) c_t \\ &= \gamma c_t \sum_{s=0}^{T-1} \left( \frac{\beta}{1+\mu} \right)^s \\ &\longrightarrow \gamma c_t \frac{1+\mu}{1+\mu-\beta} \text{ as } T \to \infty. \end{aligned}$$

By Assumption E and Sublemma 6-($iii$), it holds that

$$\begin{aligned} LHS &= m_t - \left( \prod_{j=1}^{T} \frac{1}{1+i_{t+j-1}} \frac{m_{t+j-1}}{m_{t+j}} \right) m_{t+T} \\ &= m_t - m_t \left( \prod_{j=1}^{T} \frac{1}{1+i_{t+j-1}} \right) \\ &\longrightarrow m_t \text{ as } T \to \infty. \end{aligned}$$

Thus, $m_t = \gamma c_t \frac{1+\mu}{1+\mu-\beta} = \frac{\gamma(1-\beta)}{1+\gamma} w_t \frac{1+\mu}{1+\mu-\beta}$ for every $t = 0, 1, \cdots$ . ■

As a direct consequence, we get the equilibrium price of the real money.

**Sublemma 8** *It holds that*

$$\delta_t = \frac{1+\mu-\beta}{1+\mu}, \qquad \delta_t^* = \frac{1+\mu^*-\beta}{1+\mu^*}$$

*for every* $t = 0, 1, \cdots$ .

**Proof.** Since $\delta_t m_t = \frac{\gamma(1-\beta)}{1+\gamma} w_t$, it holds, in an equilibrium, that

$$\delta_t = \frac{\gamma(1-\beta)}{1+\gamma} w_t \frac{1+\gamma}{\gamma(1-\beta)} \frac{1}{w_t} \frac{1+\mu-\beta}{1+\mu} = \frac{1+\mu-\beta}{1+\mu}.$$

■

**Proof of Lemma 1.**

$(i)$ is a direct result from Sublemma 3 and Assumption A.

$(ii)$ follows from Sublemma 5 and 8.

$(iii)$ This is a direct corollary of Sublemma 4 and 8.

$(v)$ Note that

$$w_t = \frac{A_0}{p_0} + \frac{y-\tau}{1-\beta} \text{ and } w_t^* = \frac{A_0^*}{p_0^*} + \frac{y^*-\tau^*}{1-\beta} \tag{37}$$

hold from the definition of the initial wealth, $(i)$ and Sublemma 1. The desired result follows from (36) and (37).

Since proofs of $(vi)$ and $(vii)$ are quite easy, we omit them here.

$(viii)$ By the flow budget constraint of the government (11), the equilibrium bond dynamics must follow the difference equation

$$B_t^g = \frac{1+\mu}{\beta} B_{t-1}^g + (1+\mu)^t \left[p_0(g-\tau) - \mu M_{-1}^g\right], \quad t = 1, 2, \cdots, \tag{38}$$

with

$$B_0^g = p_0 (g-\tau) - \mu M_{-1}^g + (1+i_{-1}) B_{-1}^g.$$

The solution to the difference equation (38) is given by

$$B_t^g = \left(\frac{1+\mu}{\beta}\right)^t (1+i_{-1}) B_{-1}^g + (1+\mu)^t \left[p_0(g-\tau) - \mu M_{-1}^g\right] \sum_{s=0}^{t} \beta^{-s}. \tag{39}$$

■

## 6.2 Appendix II — Proof of Lemma 2

In Appendix II, we derive the NPG condition (20).

By (39), we can easily obtain the equilibrium dynamics of the nominal debt:

$$\begin{aligned} D_t^g &= M_{t-1}^g + (1+i_{t-1}) B_{t-1}^g \\ &= (1+\mu)^t M_{-1}^g + \frac{1+\mu}{\beta} \left[ \begin{array}{c} \left(\frac{1+\mu}{\beta}\right)^{t-1} (1+i_{-1}) B_{-1}^g \\ + (1+\mu)^{t-1} \left[p_0(g-\tau) - \mu M_{-1}^g\right] \sum_{s=0}^{t-1} \beta^{-s} \end{array} \right] \\ &= (1+\mu)^t M_{-1}^g + \left(\frac{1+\mu}{\beta}\right)^t (1+i_{-1}) B_{-1}^g \\ &\quad + (1+\mu)^t \left[p_0(g-\tau) - \mu M_{-1}^g\right] \sum_{s=1}^{t} \beta^{-s} \end{aligned}$$

for $t = 0, 1, \cdots$ . Thus, the dynamics of the government's real debt can be explicitly described as

$$\begin{aligned} \frac{D_t^g}{p_t} &= \frac{D_t^g}{(1+\mu)^t p_0} \\ &= \frac{M_{-1}^g}{p_0} + \left(\frac{1}{\beta}\right)^t (1+i_{-1}) \frac{B_{-1}^g}{p_0} + \left[g - \tau - \mu \frac{M_{-1}^g}{p_0}\right] \sum_{s=1}^{t} \beta^{-s}. \end{aligned} \tag{40}$$

Using the fact that $1 + r_t = \beta^{-1}$ and the condition $p_0 > 0$, we find that the NPG condition can be rewritten as

$$\begin{aligned} NPG &\Longleftrightarrow \lim_{t\to\infty} \beta^t \frac{D_t^g}{p_t} \leq 0 \\ &\Longleftrightarrow (1+i_{-1}) \frac{B_{-1}^g}{p_0} + \left[g - \tau - \mu \frac{M_{-1}^g}{p_0}\right] \frac{1}{1-\beta} \leq 0 \\ &\Longleftrightarrow (1-\beta)(1+i_{-1}) B_{-1}^g \leq \left[p_0 (\tau - g) + \mu M_{-1}^g\right] \\ &\Longleftrightarrow \mathcal{B} \leq \gamma p_0 (\tau - g) + \theta \mathcal{M}. \end{aligned} \tag{41}$$

Thus, with the conditions in (19), we obtain

$$NPG \Longleftrightarrow \mathcal{B} \leq \theta\mathcal{M} + (\tau - g) \frac{(\mathcal{M} - \mathcal{B}\alpha)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) - \mathcal{B}(1-\alpha)\mathcal{B}^*(1-\alpha^*)}{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)}$$

$$\Longleftrightarrow \mathcal{B} \leq \theta\mathcal{M} + (\tau - g) \frac{\left[\begin{array}{c} \mathcal{M}(\mathcal{M}^* - \mathcal{B}^*\alpha^*) - \alpha(\mathcal{M}^* - \mathcal{B}^*\alpha^*) \cdot \mathcal{B} \\ -(1-\alpha)\mathcal{B}^*(1-\alpha^*) \cdot \mathcal{B} \end{array}\right]}{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)}$$

$$\begin{aligned} \Longleftrightarrow \quad & \mathcal{B} \leq \theta\mathcal{M} + (\tau - g) \frac{\mathcal{M}(\mathcal{M}^* - \mathcal{B}^*\alpha^*)}{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)} \\ & -(\tau - g) \frac{\alpha(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (1-\alpha)\mathcal{B}^*(1-\alpha^*)}{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)} \mathcal{B} \end{aligned}$$

$$\begin{aligned} \Longleftrightarrow \quad & \left[1 + (\tau - g) \frac{\alpha(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (1-\alpha)\mathcal{B}^*(1-\alpha^*)}{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)}\right] \mathcal{B} \\ \leq \quad & \theta\mathcal{M} + (\tau - g) \frac{\mathcal{M}(\mathcal{M}^* - \mathcal{B}^*\alpha^*)}{(y-\tau)(\mathcal{M}^* - \mathcal{B}^*\alpha^*) + (y^* - \tau^*)\mathcal{B}^*(1-\alpha^*)} \end{aligned}$$

$$\Longleftrightarrow \left[\begin{array}{c}(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\\+(\tau-g)\left[\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\right]\end{array}\right]\mathcal{B}$$
$$\leq \theta\mathcal{M}\left[(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\right]$$
$$+(\tau-g)\mathcal{M}(\mathcal{M}^*-\mathcal{B}^*\alpha^*)$$

$$\Longleftrightarrow \left[\begin{array}{c}(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\\+(\tau-g)\left[\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\right]\end{array}\right]\mathcal{B}$$
$$\leq \mathcal{M}\left[\begin{array}{c}(y-\tau)\theta(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\theta\mathcal{B}^*(1-\alpha^*)\\+(\tau-g)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\end{array}\right]$$

$$\Longleftrightarrow \mathcal{B}\leq\mathcal{M}\frac{\left[\begin{array}{c}(\tau-g)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y-\tau)\theta(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\\+(y^*-\tau^*)\theta\mathcal{B}^*(1-\alpha^*)\end{array}\right]}{\left[\begin{array}{c}(\tau-g)\left[\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\right]\\+(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\end{array}\right]}$$

### 6.3 Appendix III — Proof of Lemma 3

In Appendix III, we show a detailed derivation process of Lemma 3.

Let

$$\begin{aligned}\overline{\varphi} &\equiv (\tau-g)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\\&\quad+(y-\tau)\theta(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\theta\mathcal{B}^*(1-\alpha^*),\\\underline{\varphi} &\equiv (\tau-g)\{\alpha(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\mathcal{B}^*(1-\alpha^*)\}\\&\quad+(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*).\end{aligned}$$

Note the following: First, the sign of $\overline{\varphi}$ is indeterminate even if $\alpha^*$ and $\beta$ are close enough to 1, while $\underline{\varphi}>0$ by (19). Second, $1+\mu-\beta+\gamma\mu>0$ by Assumption $D'$. Third,

$$\begin{aligned}\frac{\partial\mathcal{M}}{\partial\mu} &= (1+\gamma)M_{-1}^g,\\\frac{\partial\theta}{\partial\mu} &= \frac{\gamma(1+\mu-\beta+\gamma\mu)-(1+\gamma)\gamma\mu}{(1+\mu-\beta+\gamma\mu)^2}=\frac{\gamma(1-\beta)}{(1+\mu-\beta+\gamma\mu)^2}.\end{aligned}$$

We obtain

$$\frac{\partial\varphi}{\partial\mu}>0\Longleftrightarrow\frac{\partial\mathcal{M}}{\partial\mu}\cdot\overline{\varphi}+\mathcal{M}\cdot\frac{\partial\overline{\varphi}}{\partial\mu}>0$$
$$\Longleftrightarrow(1+\gamma)M_{-1}^g\cdot\overline{\varphi}+(1+\mu-\beta+\gamma\mu)M_{-1}^g\left[\begin{array}{c}(y-\tau)(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\\+(y^*-\tau^*)\mathcal{B}^*(1-\alpha^*)\end{array}\right]\frac{\partial\theta}{\partial\mu}>0$$

$$\Longleftrightarrow (1+\gamma)\left[\begin{array}{c}(\tau-g)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+(y-\tau)\,\theta\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)\\+\left(y^*-\tau^*\right)\theta\mathcal{B}^*(1-\alpha^*)\end{array}\right]$$
$$+\left(1+\mu-\beta+\gamma\mu\right)\left[\begin{array}{c}(y-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)\\+\left(y^*-\tau^*\right)\mathcal{B}^*(1-\alpha^*)\end{array}\right]\frac{\gamma\,(1-\beta)}{\left(1+\mu-\beta+\gamma\mu\right)^2}>0$$

$$\Longleftrightarrow (1+\gamma)\left[\begin{array}{c}(\tau-g)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+(y-\tau)\,\theta\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)\\+\left(y^*-\tau^*\right)\theta\mathcal{B}^*(1-\alpha^*)\end{array}\right]$$
$$+\left[(y-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+\left(y^*-\tau^*\right)\mathcal{B}^*(1-\alpha^*)\right]\frac{\gamma\,(1-\beta)}{1+\mu-\beta+\gamma\mu}>0$$

$$\Longleftrightarrow (1+\mu-\beta+\gamma\mu)\left[\begin{array}{c}(\tau-g)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+(y-\tau)\,\theta\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)\\+\left(y^*-\tau^*\right)\theta\mathcal{B}^*(1-\alpha^*)\end{array}\right]$$
$$+\left[(y-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+\left(y^*-\tau^*\right)\mathcal{B}^*(1-\alpha^*)\right]\frac{\gamma\,(1-\beta)}{1+\gamma}>0$$

$$\begin{aligned}\Longleftrightarrow\quad &\left[\gamma\mu+\frac{\gamma\,(1-\beta)}{1+\gamma}\right]\left[(y-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+\left(y^*-\tau^*\right)\mathcal{B}^*(1-\alpha^*)\right]\\ >\quad &\left(1+\mu-\beta+\gamma\mu\right)(g-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)\end{aligned}$$

$$\Longleftrightarrow \frac{(y-\tau)\left(\mathcal{M}^*-\mathcal{B}^*\alpha^*\right)+\left(y^*-\tau^*\right)\mathcal{B}^*(1-\alpha^*)}{\mathcal{M}^*-\mathcal{B}^*\alpha^*}\cdot$$
$$\frac{1}{\left(1+\mu-\beta+\gamma\mu\right)}\left[\gamma\mu+\frac{\gamma\,(1-\beta)}{1+\gamma}\right]>g-\tau$$

### 6.4 Appendix IV — Proof of Lemma 4

In Appendix IV, we show a detailed derivation process of Lemma 4.

Note that $\partial\mathcal{M}^*/\partial\mu^*=(1+\gamma^*)\,M_{-1}^{g*}\,(>0)$. Then,

$$\begin{aligned}\frac{\partial\varphi}{\partial\mu^*}\quad &<\quad 0\\ &\Longleftrightarrow\quad \frac{\partial\overline{\varphi}}{\partial\mu^*}\cdot\underline{\varphi}<\frac{\partial\underline{\varphi}}{\partial\mu^*}\cdot\overline{\varphi}\\ &\Longleftrightarrow\quad \left[(y-\tau)\,\theta-(g-\tau)\right]\cdot\frac{\partial\mathcal{M}^*}{\partial\mu^*}\cdot\underline{\varphi}<\left[(y-\tau)-(g-\tau)\,\alpha\right]\cdot\frac{\partial\mathcal{M}^*}{\partial\mu^*}\cdot\overline{\varphi}\\ &\Longleftrightarrow\quad \left[(y-\tau)\,\theta-(g-\tau)\right]\underline{\varphi}<\left[(y-\tau)-(g-\tau)\,\alpha\right]\overline{\varphi}\end{aligned}$$

$$\Longleftrightarrow \quad [(y-\tau)\,\theta-(g-\tau)]\left[\begin{array}{c}(y-\tau)\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\,\mathcal{B}^*(1-\alpha^*)\\ -(g-\tau)\,\{\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\,\mathcal{B}^*\,(1-\alpha^*)\}\end{array}\right]$$

$$< \quad [(y-\tau)-(g-\tau)\,\alpha]\left[\begin{array}{c}(y-\tau)\,\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y^*-\tau^*)\,\theta\mathcal{B}^*(1-\alpha^*)\\ -(g-\tau)\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\end{array}\right]$$

$$\begin{array}{rl}\Longleftrightarrow & (y-\tau)^2\,\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y-\tau)\,(y^*-\tau^*)\,\theta\mathcal{B}^*(1-\alpha^*)\\ & -(y-\tau)\,(g-\tau)\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)-(y^*-\tau^*)\,(g-\tau)\,\mathcal{B}^*(1-\alpha^*)\\ & -(y-\tau)\,(g-\tau)\,\theta\,\{\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\,\mathcal{B}^*\,(1-\alpha^*)\}\\ & +(g-\tau)^2\,\{\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\,\mathcal{B}^*\,(1-\alpha^*)\}\\ < & (y-\tau)^2\,\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(y-\tau)\,(y^*-\tau^*)\,\theta\mathcal{B}^*(1-\alpha^*)\\ & -(y-\tau)\,(g-\tau)\,\alpha\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)-(y^*-\tau^*)\,(g-\tau)\,\alpha\theta\mathcal{B}^*(1-\alpha^*)\\ & -(y-\tau)\,(g-\tau)\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(g-\tau)^2\,\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\end{array}$$

$$\Longleftrightarrow \quad (g-\tau)\left[\begin{array}{c}-(y^*-\tau^*)\,\mathcal{B}^*(1-\alpha^*)\\ -(y-\tau)\,\theta\,\{\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\,\mathcal{B}^*\,(1-\alpha^*)\}\\ +(g-\tau)\,\{\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)+(1-\alpha)\,\mathcal{B}^*\,(1-\alpha^*)\}\end{array}\right]$$

$$< \quad (g-\tau)\left[\begin{array}{c}-(y-\tau)\,\alpha\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)-(y^*-\tau^*)\,\alpha\theta\mathcal{B}^*(1-\alpha^*)\\ +(g-\tau)\,\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\end{array}\right]$$

$$\Longleftrightarrow \quad (g-\tau)\left[\begin{array}{c}-(y^*-\tau^*)\,\mathcal{B}^*(1-\alpha^*)-(y-\tau)\,\alpha\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\\ -(y-\tau)\,\theta\mathcal{B}^*(1-\alpha)\,(1-\alpha^*)+(g-\tau)\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\,\alpha\\ +(g-\tau)\,\mathcal{B}^*(1-\alpha)\,(1-\alpha^*)\end{array}\right]$$

$$< \quad (g-\tau)\left[\begin{array}{c}-(y-\tau)\,\alpha\theta\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)-(y^*-\tau^*)\,\alpha\theta\mathcal{B}^*(1-\alpha^*)\\ +(g-\tau)\,\alpha\,(\mathcal{M}^*-\mathcal{B}^*\alpha^*)\end{array}\right]$$

$$\Longleftrightarrow (g-\tau)\,B_{-1}^{g*}\left[\begin{array}{c}(y-\tau)\,\theta(1-\alpha)(1-\alpha^*)+(y^*-\tau^*)\,(1-\alpha^*)\,(1-\alpha\theta)\\ -(g-\tau)\,(1-\alpha)(1-\alpha^*)\end{array}\right]>0$$

$$\Longleftrightarrow (g-\tau)\,B_{-1}^{g*}\,[(y-\tau)\,\theta(1-\alpha)+(y^*-\tau^*)\,(1-\alpha\theta)-(g-\tau)\,(1-\alpha)]>0$$

We've obtained the desired result.