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Public Pensions and Residential Choice in the Family: The Case of Parents' Possible Moving

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The Case of Parents' Possible Moving[1]

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Abstract

We examine the effects of public pensions on location patterns in a family, using a two-period model of residential choices, in which the child chooses her location in the first period, and the aged parents decide whether or not to move to their child's location in the second period. The child is altruistic toward the parents, and provides them with attention as well as financial support in two ways: income transfers and contribution to family public goods. We find that, even if the parents and child live in the same home under a certain level of public pensions, the child eventually chooses to live in a location with potential for highest earnings, where the parents would not move, as the level of public pension rises.

JEL: H41, H55, J10

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1. Introduction

When young adults seek a job after completing their education, they should choose where to live and work henceforth. While various factors affect their location choices, an important one should be the possibility of providing attention or care for their elderly parents in the future. This is because the cost of caregiving, which includes the time as well as the transportation cost, crucially depends on the distance between their own and their parents' residence. Such a factor will become more important in the location choices of young adults, as longevity proceeds and the number of potential caregivers decreases due to low fertility.

Since social security is a socialized inter-generation support scheme aiming at securing the living of the elderly, it may affect the location choices of young adults by reducing their willingness to support their aged parents. In accordance with this commonly held view, the percentage of elderly people over 65 years old living with their children has been decreasing for three decades, from 69% in 1980 to 40% in 2013 (see Figure 1), [2] with the development of the social security system in Japan.[3]

In this paper, we study the effects of public pensions on the residential choices of adult children and their parents. Adult children may provide both financial support and attention (or care) for their aged parents. The distance between them and their parents matters when they provide the latter, but it does not matter when they provide the former in the form of income transfers. Since public pensions and long-term care insurance substitute partly for the former and the latter respectively, the mechanism of public pensions for affecting residential choices in a family may be less straightforward than that of long-term care insurance. This paper, therefore, attempts to clarify how

[2] This paper does not seek to analyze factors involving the decline in the percentage of elderly people who live in the same house as their children in Japan. Many factors such as the change in industrial structure should have affected this percentage. We rather focus our attention to show that public pensions also can cause the change in geographic mobility in the family.

[3] In Japan, universal public pension (as well as universal health insurance), which covers all citizens, was introduced in 1961. The benefit payment level was improved gradually in the 1960s and the 1970s, while since the 1990s the public pension system was revised and the premium was increased in response to the aging population. In 2000, the long-term care insurance system was introduced to cover the long-term care of the elderly, because service provision under the existing health and welfare system was insufficient with the increasing number of the elderly requiring long-term care.

public pensions affect location patterns in a family and provide a rationale for the increase in the percentage of elderly living apart from their children with the development of public pensions.

Our model consists of a two-period game between the parents and the child in a family. In the first period, the child, who is a young adult and has been living in her parents' house, chooses her location and becomes employed in the labor market in the region where she lives. The child's future earnings depend on her location, and the child may choose to continue to live with her parents and work in the home region or to move to another region with better earning opportunities. In the second period, the parents age and require attention (or care). The level of attention the parents receive from the child depends on the geographical distance between the parents and the child. The child contributes to public pensions from her income and allocates the rest among her consumption of private goods, contribution to family public goods and income transfers toward her parents. The parents allocate the sum of their income (e.g. income from interest), public pension benefits and income transfers from their child between their consumption of private goods and contribution to family public goods. Incorporating family public goods into a model of location choice is a unique feature of this paper. All family members living in the same home can receive benefits from family public goods, such as houses, gardens, household appliances and housework. However, such spill-over effects almost disappear when the parents and child live apart from each other.

We assume that the child never moves in the second period because the cost is too high for professional or social reasons. Under this assumption, in the case where the child chooses to live in a distant region in the first period, the parents may have a motivation to move to the child's location. This implies that, anticipating the parents' reaction in the second period, the child decides her location strategically in the first period. The significant factors in making this decision are considered to be as follows. 1) The difference in earnings among regions. The child is more likely to move away from the parents if there is a greater potential for higher earnings away from the parents' location. 2) The level of attention the child gives the parents. The further away the child lives from her parents, the lower her attention level becomes. Therefore, the child's location choice depends on the child's preference for giving her parents attention. 3) The difference in the cost of living in terms of the distance between the parents and child. When all of them live in the same home, several types of goods serve as family public goods and thus the total expenditures of the parents and child can be relatively reduced. The child has potentially two ways of providing financial support for her parents: income transfers and contribution to the family public goods. However, the child can do this only through income transfers once she lives away from the parents.

Our main results are as follows. First, two types of equilibria can exist: one in which the parents and child live in the same home in the second period, and the other in which the parents and child live apart from each other in both periods. In the former type of equilibria, the child chooses to live apart from her parents in the first period and then the parents move to their child's location in the second period. In the latter type of equilibria, the parents do not move to their child's location in the second period. The parents' action depends on the distance between them and their child because the cost of moving is increasing in distance across. Second, if the child lives with her parents, only the child contributes to family public goods while making no cash transfers to her parents. This is because contributions to family public goods are more efficient than cash transfers from the child's point of view in that the former increases the child's consumption of family public goods at the same time, while the latter decreases the child's consumption of private goods. Third, the parents and child live separately under a high enough level of public pensions, while they may live in the same home under a lower level of public pensions. The intuition of this result is as follows. Since public pensions are compulsory intergenerational income transfers, an increase in public pensions shrinks the child's advantage of living with her parents, which is that she can provide financial support for her parents by contributing to family public goods and need not transfer income. Furthermore, an increase in public pensions lowers the child's utility because she is choosing to transfer no income toward her parents, and thus provides the child with motivation for higher earnings. This implies that the parents and child live separately in both periods, if a location where there is a potential for higher earnings is so far that the parents would not choose to move there in the second period.

Konrad *et al*. (2002) and Rainer and Siedler (2009) study the mobility pattern of two siblings who have the responsibility of providing care for their parents. Although those studies constitute a notable precursor to our analysis, the purpose is basically different: we focus on the impact of social security on location choices in a family, whereas social security and any other public policies are not within the scope of those studies. In addition, those studies consider solely attention or care as what children provide to their aged parents, and ignore any financial support. From our point of view, financial support such as income transfers and provision of family public goods by children also contributes to improve parental well-being, and should have an interaction with location choices in a family.

This paper is organized as follows. Section 2 describes the model. Section 3 derives the equilibrium of the model, and examines the effect of public pensions on the location choice in the family. Section 4 summarizes the paper.

2. Model

We consider a linear economy where the economic activity is made on a real line, and a representative family that consists of parents and an only child. The parents live and raise their child at some place that is normalized to 0.

Our model consists of two periods. In the first period, the child chooses her location $k (\geq 0)$ soon after finishing school. She is employed in the labor market in the region where she lives, and earns her income $Y(k)$ there. The child's income depends on her location and we make the following assumption: the maximum income is obtained at k^c, and the income falls as the child lives farther from k^c, where $k^c (> 0)$ represents the central business district in the linear economy. This implies that, when the child lives in the same home or locality as her parents and becomes employed in the local labor market, her income would be less than if employed at k^c: $\underset{k}{Max}\, Y(k) = Y(k^c) > Y(0)$ with $Y'(k) \geq 0$ for $k \in [0, k^c]$ (equality holds when

$k = k^c$) and $Y'(k) < 0$ for $k \in (k^c, \infty)$. We also assume $Y''(k) = 0$ for $k \in [0, k^c]$ to simplify the analysis.

In the second period, the parents retire and decide whether or not they move. We assume that the child does not move to the parents because the cost of moving is too high for professional or social reasons, as in Konrad *et al*. (2002). In the parents' decision on location p, the following two factors should be important. On the one hand, the parents are old and need attention (or care) in the second period, and the level of attention the child gives to the parents depends on the distance between the parents and child because longer travel time means a greater cost of the visit. Therefore, denoting the distance between the parents and child (the distance between points p and k) as $\delta(p, k)$, the level of attention is provided as $a = a(\delta(p, k))$ with $da(\delta)/d\delta < 0$. On the other hand, moving costs the parents. They have lived in the same place for a long time and have built up a social network of friends in their local area. If they move, they may lose their local friendship ties. In addition, it may take time and effort to become accustomed to their new environment in a region with different custom and culture. We denote the parents' cost for moving as $\eta(p) = \eta p$, which is assumed to depend on the distance between the new location p and the present location (point 0). This is because the longer distance from their local friends implies the higher costs of maintaining social contacts. Also, the difference in custom and culture may be greater, if they move over longer distances.

We consider two types of goods: private goods and family public goods. The benefits from family public goods spill over to all family members. We assume that, as long as family members live in the same home, the family public goods have the property of pure public goods. The supply of family public goods is thus equal to the sum of contributions made by the parents and child, g_p and g_k, if $\delta = 0$. On the other hand, even when the parents and child do not live in the same home ($\delta > 0$), the property of public goods still exists to some extent for several types of family public goods if they live in the same neighborhood and visit each other's home very frequently. However, such spill-over effects become smaller as the distance between parents and child becomes greater, eventually disappearing at a certain distance, which is denoted as

$\bar{\delta}$. Therefore, the levels of family public goods consumed by the parents and child, G_p and G_k, are determined as follows:

(1) $$G_p = g_p + \gamma(\delta) g_k,$$

(2) $$G_k = g_k + \gamma(\delta) g_p,$$

where $\gamma(\delta)$ indicates the magnitude of spill-over effects of the child's (parents') contribution to family public goods. It is assumed that $0 \le \gamma(\delta) \le 1$, $\gamma(0) = 1$ and $\gamma(\delta) = 0$ for $\delta \ge \bar{\delta}(>0)$.

After the parents choose their location p in the second period, they allocate the sum of their income (e.g., income from interest) Y_p, public pension benefits T_p and income transfers from their child $\pi(\ge 0)$ between their consumption of private goods C_p and contribution to family public goods g_p. The budget constraint for the parents is thus given by

(3) $$C_p = Y_p - g_p + \pi + T_p.$$

The child contributes T_k to public pensions from her income $Y_k(k)$ and allocates the rest among her consumption of private goods C_k, contribution to family public goods g_k, and income transfers toward her parents. The budget constraint for the child is thus given by

(4) $$C_k = Y_k(k) - g_k - \pi - T_k.$$

Assuming that all families are identical, we have $T_k = T_p = T$ under a pay-as-you-go public pension system.

The child is altruistic toward her parents, and her utility function is given by

(5) $$U_k = \log C_k + \alpha \log G_k + v_k(a(\delta)) + \rho U_p,$$

where ρ $(0 < \rho < 1)$ is the weight attached to the parents' utility U_p, and $\alpha > 0$ is assumed. On the other hand, the parents are non-altruistic and their utility function is given by

(6) $$U_p = \log C_p + \alpha \log G_p + v_p(a(\delta)) - \eta p.$$

According to Bernhaim *et al*. (1985), we assume that both the parents' and child's utility derived from attention, $v_p(\cdot)$ and $v_k(\cdot)$, first increase and then decrease in a ($v_p''(\cdot) < 0$ and $v_k''(\cdot) < 0$), and that the parents' utility $v_p(\cdot)$ always increases when the child's utility $v_k(\cdot)$ does not decrease in a ($\arg\max_a v_k(a) \le \arg\max_a v_p(a)$).

Also, we make the following assumption on $a(0)$, the level of attention when the parents and child live in the same home:

(7) $$\arg\max_a v_k(a) \le a(0) \le \arg\max_a [v_k(a) + \rho v_p(a)],$$

which implies that the child's *private* utility of attention is decreasing while the child's *total* utility (including the altruistic term) is increasing when $\delta = 0$ ($v_k'(a(0)) \le 0$ and $v_k'(a(0)) + \rho v_p'(a(0)) \ge 0$). From (7), we also find that the parents' utility is increasing when $\delta = 0$ ($v_p{}'(a(0)) \ge 0$). From the assumptions made above, $v_k'(a(0)) + \rho v_p'(a(0)) \ge 0$, $v_k''(a(\delta)) + \rho v_p''(a(\delta)) < 0$ and $a(\delta) < a(0)$ for $\delta > 0$, we have $v_k'(a(\delta)) + \rho v_p'(a(\delta)) > 0$ for $\delta > 0$.

The timing of the game is as follows. In the first period, (1) the child chooses her location k. In the second period, (2) the parents choose their location p; (3) the parents choose their contribution to family public goods g_p; (4) the child chooses her consumption of private goods C_k, her contribution to family public goods g_k, and income transfers toward her parent π. (As a result, the parents' consumption of private goods C_p is determined.)

3. Effect of public pensions on location choice

In this section, we derive the subgame perfect equilibrium of the model presented in the previous section, and examine the effect of public pensions on the parents' and child's location choice.

3.1 Parents' and child's contribution to family public goods

In the fourth stage, given the parents' contribution to family public goods g_p, the parents' location p, her own location k, and the contribution to public pensions T, the child chooses the contribution to family public goods g_k and income transfers to the parents $\pi\ (\geq 0)$ so as to maximize her utility (5). The first-order conditions for maximization are[4]

(8) $$-\frac{1}{Y_k(k)-g_k-\pi-T}+\frac{\rho}{Y_p-g_p+\pi+T}\leq 0 \text{ (equality holds if } \pi>0\text{)},$$

(9) $$-\frac{1}{Y_k(k)-g_k-\pi-T}+\frac{\alpha}{g_k+\gamma(\delta)g_p}+\frac{\rho\alpha\gamma(\delta)}{g_p+\gamma(\delta)g_k}=0.$$

The child's reaction functions are derived from (8) and (9), and defined as

(10) $$\pi=\pi(g_p,\ p,\ k,\ T)=\begin{cases}\pi^+(g_p,\ p,\ k,\ T) & \text{(if (8) holds with equality)},\\ 0 & \text{(if (8) holds with strict inequality)},\end{cases}$$

(11) $$g_k=g_k(g_p,\ p,\ k,\ T)=\begin{cases}g_k^+(g_p,\ p,\ k,\ T) & \text{(if (8) holds with equality)},\\ g_k^0(g_p,\ p,\ k,\ T) & \text{(if (8) holds with strict inequality)},\end{cases}$$

with

(12) $$\frac{\partial\pi^+}{\partial g_p}=1+\frac{(1-\gamma)}{D}\frac{\alpha}{C_k^2}\left(-\frac{1}{G_k^2}+\frac{\rho\gamma}{G_p^2}\right)>0,$$

(13) $$\frac{\partial g_k^+}{\partial g_p}=-1+\frac{\alpha(1-\gamma)}{D}\left(\frac{1}{C_k^2}+\frac{\rho}{C_p^2}\right)\left(\frac{1}{G_k^2}-\frac{\rho\gamma}{G_p^2}\right)<0,$$

(14) $$\frac{\partial g_k^0}{\partial g_p}=-\frac{\alpha\gamma[(1/G_k^2)+(\rho/G_p^2)]}{(1/C_k^2)+\alpha[(1/G_k^2)+(\rho\gamma^2/G_p^2)]}<0,$$

[4] For the following reasons, g_k always takes a positive value. First, since we consider a child supporting her parents financially, we assume away the case where both π and g_k are zero. Second, even if we consider the non-negativity constraint on g_k explicitly, it cannot take a corner solution when $\pi>0$. This is proved as follows. Suppose that $\pi>0$ and the non-negativity constraint on g_k is binding. From (8) with equality and the first-order condition with respect to g_k ($-1/(Y_k(0)-\pi-T)+\alpha(1+\rho)/g_p<0$), we have

$$\frac{-\rho}{Y_p-g_p+\pi+T}+\frac{\alpha(1+\rho)}{g_p}<0.$$

This is equal to the marginal utility of g_p (the left-hand side of (15)) because $\partial\pi/\partial g_p=1/(1+\rho)$ is obtained from (8) if $\pi>0$ and $g_k=0$. We thus have $g_p=0$. However, this implies $\alpha(1+\rho)/g_p=\infty$, which contradicts the first-order condition with respect to g_k.

where[5]

$$D = \frac{1}{C_k^2}\left(\frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2}\right) + \frac{\rho}{C_p^2}\left(\frac{1}{C_k^2} + \frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2}\right) > 0.$$

In the third stage, given p, k and T, taking the child's reaction functions (10) and (11) into account, the parents choose the contribution to family public goods g_p so as to maximize their utility (6). The first-order condition for maximization is

(15)

$$\frac{1}{Y_p - g_p + \pi + T}\left(\frac{\partial \pi^+}{\partial g^p} - 1\right) + \frac{\alpha}{g_p + \gamma(\delta) g_k}\left(1 + \gamma(\delta)\frac{\partial g_k^+}{\partial g_p}\right) \le 0 \text{ (if (8) holds with equality),}$$

(16)

$$-\frac{1}{Y_p - g_p + T} + \frac{\alpha}{g_p + \gamma(\delta) g_k}\left(1 + \gamma(\delta)\frac{\partial g_k^0}{\partial g_p}\right) \le 0 \text{ (if (8) holds with strict inequality).}$$

We define the parents' reaction function derived from (15) and (16) as

$$\text{(17)} \qquad g_p = g_p(p,\ k,\ T) = \begin{cases} g_p^+(p,\ k,\ T) & \text{(if (8) holds with equality),} \\ g_p^0(p,\ k,\ T) & \text{(if (8) holds with strict inequality).} \end{cases}$$

From (12)-(14), if the child lives with her parents ($k = p$), we have $\gamma = 1$, implying $\partial \pi^+ / \partial g_p = 1$, $\partial g_k^+ / \partial g_p = -1$ and

$$\text{(18)} \qquad \frac{\partial g_k^0}{\partial g_p} = -(1-\theta) < 0,$$

where

$$\text{(19)} \qquad \theta = \frac{\alpha(1+\rho)}{1+\alpha(1+\rho)} \quad (0 < \theta < 1).$$

This implies that, if (8) holds with equality (namely, $\pi > 0$ or $\pi = 0$ as the interior solution), the left-hand side of (15) is zero for any value of g_p, so that indeterminacy arises for g_p. It follows from (8) and (9) that the indeterminacy of g_p entails the indeterminacy of π and g_k. This result is similar to that obtained in Cornes, Itaya and Tanaka (2012). The following proposition provides a sufficient condition under which

[5] The derivation of (12)-(14) and (18) is shown in the Appendix.

we have $\pi = 0$ and $g_p = 0$ as the corner solution, and the indeterminacy does not arise in the equilibrium of the subgame beginning at the third stage, given that the child lives with the parents.

Proposition 1. Given $k = p$. If $\rho(1-\theta) < (Y_p + T)/(Y_k(k) - T) < 1/\alpha$, then $\pi = 0$ and $g_p = 0$.

Proof:

Consider the child's choice on π in the fourth stage. The first-order condition (8) implies that, given $g_p = 0$, we have $\pi = 0$ if

$$\rho < \frac{Y_p + T}{Y_k(k) - g_k - T}. \tag{20}$$

Substituting $\gamma(0) = 1$, $g_p = 0$ and $\pi = 0$ into (9) yields

$$g_k = \theta[Y_k(k) - T]. \tag{21}$$

Substituting (21) into (20) yields

$$\rho(1-\theta) < \frac{Y_p + T}{Y_k(k) - T}. \tag{22}$$

Given $g_p = 0$, we have $\pi = 0$ if (22) holds.

Next, we examine the parents' choice on g_p in the third stage when (22) holds. We define $\hat{g}_p$ as the level of the parents' contribution to family public goods such that income transfers π are operative for $g_p > \hat{g}_p$.[6] The marginal utility of g_p for $g_p > \hat{g}_p$ is given by

$$\left.\frac{dU_p}{dg_p}\right|_{g_p > \hat{g}_p} = \frac{1}{Y_p - g_p + \pi + T}\left(\frac{\partial \pi^+}{\partial g^p} - 1\right) + \frac{\alpha}{g_p + g_k}\left(1 + \frac{\partial g_k^+}{\partial g_p}\right). \tag{23}$$

Since we have $\partial \pi^+ / \partial g_p = 1$ and $\partial g_k^+ / \partial g_p = -1$ if $k = p$ as shown above, (23) is

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[6] The child's marginal utility of π is increasing in g_p:

$$\frac{\partial}{\partial g_p}\left(\frac{\partial U_k}{\partial \pi}\right) = -\frac{-(-1)}{C_k^2}\frac{\partial g_k^0}{\partial g_p} + \frac{\rho}{C_p^2} = \frac{1-\theta}{C_k^2} + \frac{\rho}{C_p^2} > 0.$$

Therefore, the child chooses positive π under a sufficiently large level of g_p.

equal to zero. On the other hand, since $\pi = 0$ for $g_p \leq \hat{g}_p$, we have

(24) $$\left.\frac{dU_p}{dg_p}\right|_{0 \leq g_p \leq \hat{g}_p} = -\frac{1}{Y_p - g_p + T} + \frac{\alpha}{g_p + g_k}\left(1 + \frac{\partial g_k^0}{\partial g_p}\right).$$

Substituting $g_k = -(1-\theta)g_p + \theta(Y_k(k) - T)$, which is obtained from (9) with $\delta = 0$, and (18) into (24) yields

(25) $$\left.\frac{dU_p}{dg_p}\right|_{0 \leq g_p \leq \hat{g}_p} = -\frac{1}{Y_p - g_p + T} + \frac{\alpha}{Y_k(k) + g_p - T},$$

which is negative,[7] if

(26) $$\frac{Y_p + T}{Y_k(k) - T} < \frac{1}{\alpha}.$$

Given $k = p$, therefore, U_p is maximized at $g_p = 0$, if (26) holds.

The above argument shows that, if (22) and (26) are simultaneously satisfied, we have $g_p = 0$ and $\pi = 0$. □

Proposition 1 suggests that, when the child lives with the parents in the same home, both income transfers from the child to the parents and the parents' contribution to family public goods are zero, given reasonable parameter values. For example, under $\rho = 0.6$ and $\alpha = 1$, we have $g_p = 0$ and $\pi = 0$ if $0.6/2.6 (\approx 0.23) \leq (Y_p + T)/(Y_k(k) - T) < 1$. The ratio of disposable income of the retired generation to that of the working generation is likely to take a value within this range. In the analysis below, we assume that the sufficient condition in Proposition 1 is satisfied.

3.2 Parents' location choice

We now examine the parents' location choice in the second stage. Given the child's location k, they choose their location p so as to maximize the utility function

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[7] We have $(Y_p - g_p + T)/(Y_k(0) + g_p - T) < (Y_p + T)/(Y_k(0) - T)$. This implies that (25) is negative if (26) holds.

(6) subject to the reaction functions (10), (11) and (17). If the child is living with the parents ($k=0$), the parental utility is maximized at $p=0$, implying that the parents need not to move.

However, if the child is living apart from the parents ($k>0$), the parental choice on whether or not to move is more complicated. We consider the case where $k>\bar{\delta}$, namely, the distance between parents and child is great enough for the spill-over effects of family public goods to vanish.[8] Obviously, since the cost of moving is increasing in the distance between, the parents never move to a location which is further than k from their present location, implying that the parents location p must be in $[0, k]$.

We examine the change in each term of the parental utility function (6) as p changes.

(1) Change in cost for moving ηp: It increases proportionally as p rises from 0 to k.

(2) Change in utility from attention $v_p(a(\delta))$: A rise in p shortens the distance between the parents and the child, and increases attention and $v_p(a(\delta))$, which is maximized at $p=k$.

(3) Change in utility from consumption $\log C_p + \alpha \log G_p (\equiv \tilde{U}_p)$: When the parents and the child live in the same home ($p=k$), we have $C_p = Y_p + T$ and $G_p = g_k$ ($\because g_p = \pi = 0$) under the sufficient condition in Proposition 1.

Suppose that p decreases gradually from k. As long as $g_p = \pi = 0$ holds, C_p does not change and G_p $(=\gamma(\delta) g_k)$ changes only through the change in $\gamma(\delta)$ with a decrease in p, because (9) implies that g_k does not depend on $\gamma(\delta)$ when $g_p = 0$. Hence G_p decreases as p decreases from k. Since a decrease in $\gamma(\delta)$ raises the marginal utility of g_p (the left-hand side of (16)),[9] g_p becomes positive when p decreases to a certain level, which is denoted by $\tilde{p}$.[10] For $\tilde{p} < p \le k$, the above

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[8] Even when $k \le \bar{\delta}$, the result obtained below could be maintained. See footnote 14.

[9] Substituting $g_p = 0$ into the left-hand side of (16) and differentiating it with respect to γ yields

$$-\frac{\alpha}{(\gamma g_k)^2} g_k \left(1+\gamma \frac{\partial g_k^0}{\partial g_p}\right) + \frac{\alpha}{\gamma g_k} \frac{\partial g_k^0}{\partial g_p} = -\frac{\alpha}{\gamma^2 g_k} < 0 .$$

[10] When $g_p = 0$, the marginal utility of π (the left-hand side of (8)) is constant. This implies $\pi = 0$

discussion implies that $\tilde{U}_p$ decreases with a decrease in p ($d\tilde{U}_p / dp > 0$).

If the marginal utility of π (the left-hand side of (8)) rises as p decreases from $\tilde{p}$, π becomes positive at a certain level of p, which we denote as $\hat{p}$.[11] When p decreases further and reaches $k-\overline{\delta}$, the spill-over effect of family public goods disappears. While the sign of $d\tilde{U}_p / dp$ is indeterminate for $k-\overline{\delta} < p \le \tilde{p}$,[12] we have $d\tilde{U}_p / dp = 0$ for $0 \le p \le k-\overline{\delta}$ because the distance between the parents and the child does not affect C_p and G_p (see also Figure 1).

Based on the above analysis, we now examine the change in the parental utility U_p with a change in p. To simplify the analysis, we make the following assumptions:

Assumption 1. $d^2\tilde{U}_p / (dp)^2 \ge 0$ and $d^2 v_p(a(\delta)) / (dp)^2 \ge 0$ for $k-\overline{\delta} \le p \le k$

Assumption 2. $\eta > dv_p(a(\delta)) / dp$ for $0 \le p \le k-\overline{\delta}$

In Assumption 1, the sign of $d^2\tilde{U}_p / (dp)^2$ crucially depends on the functional form of $\gamma(\delta)$. The sufficient condition for $d^2\tilde{U}_p / (dp)^2 \ge 0$ is shown in Appendix. For $d^2 v_p(a(\delta)) / (dp)^2 \ge 0$ to be satisfied, we require $d^2 a(\delta) / (dp)^2 > 0$, which implies that the increase in attention by a marginal decrease in the distance between the parents and the child becomes greater as the original distance is smaller. This is also related to Assumption 2. When the distance between the patents and the child is great enough, its marginal decrease hardly affects the attention level, implying that $dv_p(a(\delta)) / dp$ is smaller than η.

For $0 \le p \le k-\overline{\delta}$, since $d\tilde{U}_p / dp = 0$, we have $dU_p / dp < 0$ under Assumption 2. For $k-\overline{\delta} < p \le k$, the sign of dU_p / dp is indeterminate in general.[13] From

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for $p \ge \tilde{p}$.

[11] We cannot exclude the possibility that the marginal utility of π falls as p decreases. If this is the case, we have $\pi = 0$ for any p. We assume that the marginal utility of π rises and reaches zero at $\hat{p}$, where the spill-over effect of family public goods still exist, because this paper is focusing on the case where the child supports her parents financially.

[12] See Appendix.

[13] For $k-\overline{\delta} < p \le \tilde{p}$, the sign of $d\tilde{U}_p / dp$ is indeterminate, causing the indeterminacy of the sign of dU_p / dp. On the other hand, for $\tilde{p} < p \le k$, we have $d\tilde{U}_p / dp > 0$. Hence, if the sum of $d\tilde{U}_p / dp$

Assumption 1, however, we have $d^2U_p/(dp)^2 \geq 0$, which implies that, once $d\tilde{U}_p/dp + dv_p(a(\delta))/dp$ dominates η at some value of p, $dU_p/dp > 0$ holds for any greater values of p. Figure 2 shows the graph of U_p, which implies that we have $p=0$ or $p=k$, the parents remain at their present location or move to the child's location (if they move), in the equilibrium.[14]

We next examine the effect of the child's location on the parents' location choice. To do so, we compare the level of parental utility when the parents move to the child's location ($p=k$) with that when the parents do not move ($p=0$), given k.

The parental utility with $p=k$ is given by

$$\begin{aligned} U_p\Big|_{p=k} &= \log(Y_p+T) + \alpha \log g_k + v_p(a(0)) - \eta p \\ &= \log(Y_p+T) + \alpha \log[\theta(Y_k - T)] + v_p(a(0)) - \eta p. \end{aligned}$$

We examine the change in each term of the parental utility function caused by a change in $k\ (=p)$.

(1) Change in cost for moving ηp: It increases proportionally as k rises.

(2) Utility from attention $v_p(a(0))$ does not depend on k because the parents and child live in a same residence.

(3) Change in utility from consumption $\log C_p + \alpha \log G_p (\equiv \tilde{U}_p)$: As k rises, the child's income increases. This results in an increase in the child's contribution to family public goods and in the parents' consumption of family public goods.

The change in $U_p\big|_{p=k}$ is given by

(27) $$\left.\frac{dU_p}{dk}\right|_{p=k} = \alpha \frac{Y_k'}{Y_k - T} - \eta ,$$

the sign of which is indeterminate, depending on the relative magnitude between the increase in the cost for moving and the increase in consumption of family public goods

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and $dv_p(a(\delta))/dp\ (>0)$ dominates η, we have $dU_p/dp > 0$.

[14] In the case of $k \leq \bar{\delta}$, since the region $0 \leq p \leq k - \bar{\delta}$ disappears, the analysis on the region $k - \bar{\delta} < p \leq k$ in the case of $k > \bar{\delta}$ can be applied. Under Assumption 1, we have $p=0$ or $p=k$ in the equilibrium also in this case.

in utility term. Differentiating (27) with respect to k yields

(28) $$\left.\frac{d^2U_p}{(dk)^2}\right|_{p=k} = -\alpha\frac{(Y_k')^2}{(Y_k - T)^2} < 0,$$

implying that $U_p\big|_{p=k}$ is concave in k.

On the other hand, the parental utility with $p = 0$ is given by

$$U_p\big|_{p=0} = \log(Y_p - g_p + \pi + T) + \alpha\log(g_p + \gamma(\delta)g_k) + v_p(a(\delta)),$$

in which the cost of moving disappears because the parents do not move. We examine the change in each term of the parental utility function caused by a change in k.

(1) Change in utility from attention $v_p(a(\delta))$: An increase in k implies that the distance between the parents and the child becomes longer, and thus decreases the level of attention.

(2) Change in utility from consumption $\log C_p + \alpha\log G_p (\equiv \tilde{U}_p)$: When k is zero or close enough to zero, we have $C_p = Y_p + T$ and $G_p = g_k$ ($\because g_p = \pi = 0$) under the sufficient condition in Proposition 1. With an increase in k, C_p does not change and G_p $(= \gamma(\delta)g_k)$ changes only through the change in $\gamma(\delta)$. Since (9) implies that g_k does not depend on $\gamma(\delta)$ when $g_p = 0$, G_p decreases as k increases. When k increases to a certain level, which is denoted by $\tilde{k}$, g_p becomes positive because a decrease in $\gamma(\delta)$ raises the marginal utility of g_p (the left-hand side of (16)). Hence, we have $d\tilde{U}_p / dk < 0$ for $0 \le k \le \tilde{k}$.

If the marginal utility of π (the left-hand side of (8)) rises as k increases from $\tilde{k}$, π becomes positive at a certain level of k, which we denote as $\hat{k}$. When k increases further and reaches $\bar{\delta}$, the spill-over effect of family public goods disappears. While the sign of $d\tilde{U}_p / dk$ is indeterminate for $\tilde{k} < k \le \bar{\delta}$, we have $d\tilde{U}_p / dk > 0$ for $\bar{\delta} < k$ (see also Figure 1).[15] When k is greater than $\bar{\delta}$, an increase in k does not have a negative effect on $\tilde{U}_p$ through family public goods, which no longer exist. On the other hand, the child's income rises as k increases, causing an increase in the

[15] See (A.31) and (A.33) in Appendix.

income transfers toward the parents, which has a positive effect on $\tilde{U}_p$.

The above analysis suggests the following result on the change in the parental utility $U_p\big|_{p=0}$ with a change in k: for $0 \le k \le \tilde{k}$, $dU_p\big|_{p=0}/dk < 0$; for $\tilde{k} < k \le \overline{\delta}$, the sign of $dU_p\big|_{p=0}/dk$ is indeterminate because that of $d\tilde{U}_p/dk$ is indeterminate; for $\overline{\delta} < k$, we have

$$\left.\frac{dU_p}{dp}\right|_{p=0} = \frac{1}{C_p}\frac{\partial \pi}{\partial k} + v_p'(a)\cdot a'(k),$$

the sign of which is indeterminate because the first term is positive while the second term is negative.

We now compare $U_p\big|_{p=k}$ to $U_p\big|_{p=0}$ under each level of k. When $k=0$, namely, the child lives in her parents' home, we have $U_p\big|_{p=k} = U_p\big|_{p=0}$. For $0 < k \le \tilde{k}$, $dU_p\big|_{p=0}/dk < 0$ implies $U_p\big|_{p=k} > U_p\big|_{p=0}$, if (27) is positive (the increase in consumption of family public goods dominates the increase in the cost for moving in utility term), or greater than $dU_p\big|_{p=0}/dk$ even if it is negative. For $k > \tilde{k}$, however, the relative magnitude between $U_p\big|_{p=k}$ and $U_p\big|_{p=0}$ is ambiguous, because neither of signs of $dU_p\big|_{p=k}/dk$ and $dU_p\big|_{p=0}/dk$ is indeterminate. To obtain a clear result, we make the following assumptions:

Assumption 3. $d^2\tilde{U}_p\big|_{p=k}/(dk)^2 - d^2\tilde{U}_p\big|_{p=0}/(dk)^2 \le 0$ and $d^2 v_p(a(\delta))/(dk)^2 \le 0$

Assumption 4. $U_p\big|_{p=0} > U_p\big|_{p=k}$ for $k = k^c$

The sufficient condition for $d^2\tilde{U}_p\big|_{p=k}/(dk)^2 - d^2\tilde{U}_p\big|_{p=0}/(dk)^2 \le 0$ in Assumption 3 is shown in the Appendix. $d^2v_p(a(\delta))/(dk)^2 \le 0$ in Assumption 3 has an implication basically the same as $d^2v_p(a(\delta))/(dp)^2 \ge 0$ in Assumption 1. Noting that the parents' cost for moving depends positively on the moving distance, Assumption 4 implies that k^c is so far and the cost for moving there is so high that the parents would not choose to move to k^c. Under Assumptions 3 and 4, there exists $k_T \in [0,\, k^c)$ such that $U_p\big|_{p=0} = U_p\big|_{p=k}$ for $k = k_T$, as shown in Figure 3. Hence, we have $p = k$ for $0 \le k \le k_T$, and $p = 0$ for $k_T < k \le k^c$.

3.3 Child's location choice

We now examine the child's location choice in the first stage. Anticipating the parents' location choice in the second stage, the child chooses $k\ (\ge 0)$ so as to maximize the utility function (5). As shown in the previous sub-section, the parents move to the child's location ($p = k$) if $0 \le k \le k_T$, but continue to reside in their present location if $k_T < k \le k^c$.

The child's utility in the case of living with her parents is given by

$$\begin{aligned} U_k\big|_{k=p} &= \log(Y_k(k) - T - g_k) + \alpha \log g_k + v_k(a(0)) + \rho U_p\big|_{p=k} \\ &= \log[(1-\theta)(Y_k(k) - T)] + \alpha \log[\theta(Y_k(k) - T)] + v_k(a(0)) + \rho U_p\big|_{p=k} . \end{aligned}$$

Differentiating this equation with respect to k yields

(29) $$\left.\frac{dU_k}{dk}\right|_{k=p} = (1+\alpha)\frac{Y_k'}{Y_k - T} + \rho \left.\frac{dU_p}{dk}\right|_{p=k} .$$

In (29), while the sign of the second term is indeterminate as mentioned in the previous sub-section, the first term is positive, implying that $dU_k\big|_{k=p}/dk > 0$ if

$dU_p\big|_{p=k}/dk=0$, and $dU_p\big|_{p=k}/dk<0$ if $dU_k\big|_{k=p}/dk=0$. Furthermore, differentiating (29) with respect to k yields

$$\left.\frac{d^2U_p}{(dk)^2}\right|_{k=p} = -\frac{(1+\alpha)(Y_k')^2}{(Y_k-T)^2}+\rho\left.\frac{d^2U_p}{(dk)^2}\right|_{p=k},$$

which is negative from (28). Denoting k which maximizes $U_k\big|_{k=p}$ as k^{**} and k which maximizes $U_p\big|_{p=k}$ as p^{**}, the above discussion suggests $p^{**}<k^{**}$. We show $U_k\big|_{k=p}$, $U_p\big|_{p=k}$ and $U_p\big|_{p=0}$ in Figure 3, where we assume $k_T \le k^{**}$.[16]

To examine the child's location choice, we consider the change in the child's utility U_k as k changes, dividing the range of k into (i) $k \le k_T$ and (ii) $k > k_T$.

(i) $k \le k_T$: Since the parents move to the child's location ($p=k$) as shown in the previous sub-section, U_k is maximized at k_T in this range.

(ii) $k > k_T$: The parents do not move and remain in their present location. Therefore, the child's utility in this range is given by

$$U_k\big|_{k>k_T} = \log(Y_k(k)-T-g_k-\pi)+\alpha\log g_k + v_k(a(k))+\rho U_p\big|_{p=0}.$$

Differentiating this equation with respect to k and using the envelope theorem yields

$$\left.\frac{dU_k}{dk}\right|_{k>k_T} = \frac{Y_k'(k)}{C_k}+[v_k'(a)+\rho v_p'(a)]a'(k), \tag{30}$$

whose sign is indeterminate in general, because the first term is positive while the second term is negative. However, if k_T is great enough, $a'(k)$ is likely to be small enough to make (30) positive. This assumption is basically the same as Assumption 2, implying that, when the distance between the patents and the child is long enough, its marginal change hardly affects the attention level. Under this assumption, U_k is maximized at k^c in this range.

Since the above discussion suggests that U_k jumps at k_T, we next examine

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[16] See footnote 16 for the case where $k_T > k^{**}$.

whether U_k jumps upward or downward there. The child's utility when $k = k_T$ (the parents move to the child's location) is given by

$$(31) \qquad U_k\big|_{k=k_T} = \log(Y_k(k_T) - T - g_k\big|_{k=k_T}) + \alpha \log g_k\big|_{k=k_T} + v_k(a(0)) + \rho U_p\big|_{p=k_T}.$$

The child's utility when $k = k_T + \lim_{\varepsilon \to 0} \varepsilon$ (the parents do not move) is given by

$$(32) U_k\big|_{k=k_T+\varepsilon} = \log(Y_k(k_T+\varepsilon) - T - g_k\big|_{k=k_T+\varepsilon} - \pi\big|_{k=k_T+\varepsilon}) + \alpha \log g_k\big|_{k=k_T+\varepsilon} + v_k(a(k_T)) + \rho U_p\big|_{p=0}.$$

The relative magnitude between $U_k\big|_{k=k_T}$ and $U_k\big|_{k=k_T+\varepsilon}$ is determined by that between the child's private utilities (the first three terms) in (31) and (32), because we have $U_p\big|_{p=k_T} = U_p\big|_{p=0}$ from the definition of k_T. Comparing the second term of (31) to that of (32), we have $g_k\big|_{k=k_T} > g_k\big|_{k=k_T+\varepsilon}$,[17] implying that the second term of (31) is greater. However, the indeterminacy in the relative magnitude of the first and third terms between (31) and (32) makes the relative magnitude between $U_k\big|_{k=k_T}$ and $U_k\big|_{k=k_T+\varepsilon}$ indeterminate. The determination of the child's location becomes conditional as follows.[18]

Lemma 1. (1) If $U_k\big|_{k=k_T} < U_k\big|_{k=k_T+\varepsilon}$, $k = k^c$.

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[17] When $k = k_T + \lim_{\varepsilon \to 0} \varepsilon$, the parents remain at $p = 0$ and thus spill-over effects of g_k do not work. This implies that the marginal benefit of g_k is greater for $k = k_T$ than for $k = k_T + \lim_{\varepsilon \to 0} \varepsilon$. A formal proof of $g_k\big|_{k=k_T} > g_k\big|_{k=k_T+\varepsilon}$ is provided as follows. From (9), we have $[1+(1+\rho)\alpha]g_k\big|_{k=k_T} / (1+\rho)\alpha = Y_k(k_T) - T$ and $(1+\alpha) g_k\big|_{k=k_T+\varepsilon} / \alpha = Y_k(k_T) - \pi - T$. These equations imply that $[1+(1+\rho)\alpha]g_k\big|_{k=k_T} / (1+\rho)\alpha > (1+\alpha) g_k\big|_{k=k_T+\varepsilon} / \alpha$. Since $[1+(1+\rho)\alpha]/(1+\rho) < 1+\alpha$, we have $g_k\big|_{k=k_T} > g_k\big|_{k=k_T+\varepsilon}$.

[18] In the case where $k_T > k^{**}$, we have $k = k^c$ or $k = k^{**}$, depending on the relative magnitude between $U_k\big|_{k=k^{**}}$ and $U_k\big|_{k=k_C}$. Proposition 2 is maintained even in this case.

(2a) If $U_k\big|_{k=k_T} > U_k\big|_{k=k_T+\varepsilon}$ and $U_k\big|_{k=k_T} < U_k\big|_{k=k_C}$, $k = k^c$.

(2b) If $U_k\big|_{k=k_T} > U_k\big|_{k=k_T+\varepsilon}$ and $U_k\big|_{k=k_T} > U_k\big|_{k=k_C}$, $k = k_T$.

While the parents and child live apart in the case of (1) or (2a), they live in the same home in the case of (2b).

3.4 Public pensions and family location

We next examine the effect of public pensions on the parents' and child's location choices. Suppose that, given an arbitrary level of public pensions, the sufficient conditions in (2b) hold, and the parents and child live in the same home ($k = p = k_T$). We examine the change in the child's utility when the level of public pensions increases marginally. Differentiating (31) with respect to T and using the envelope theorem yields

(33) $$\frac{dU_k\big|_{k=k_T}}{dT} = -\frac{1}{Y_k(k) - g_k - \pi - T} + \frac{\rho}{Y_p - g_p + \pi + T}.$$

Since Proposition 1 suggests that we have $\pi = 0$ when $k = p$, (33) is negative from (8). On the other hand, $U_k\big|_{k=k_C}$ is not affected by public pensions, because, when $k = k^c$, we have $\pi > 0$ and thus Ricardian equivalence holds. As shown in Figure 4, therefore, when T rises and reaches a certain level, $\hat{T}$, $U_k\big|_{k=k_T}$ is dominated by $U_k\big|_{k=k_C}$, implying that the child chooses to live at k^c apart from her parent.

Proposition 2. Suppose that, given an arbitrary level of public pensions, the parents move to the child's location, and the parents and child live in the same home. If the level of public pensions rises and reaches $\hat{T}$, the child chooses to live at $k = k^c$, where her income is maximized, and the parents do not move to the child's location.

4. Conclusion

This paper attempted to examine the relevance of a commonly held view that the welfare state or social security tends to loosen family bonds, in other words, to decrease the attention or care which children provide to their parents. For this purpose, we explicitly considered the location choice in the family because the feasible level of attention should be subject to the distance between the child's and the parent's residence. In this analysis, financial support from the child to the parents in two ways, income transfers and provision of family public goods, also played a crucial role.

The main results obtained in this paper are as follows. First, two types of equilibria can exist: one in which the parents and child live in the same home in the second period, and the other in which the parents and child live apart from each other in both periods. Second, if the parents and the child live in the same home, the child transfers no income to the parents while paying everything for family public goods under a plausible condition. Third, even if the parents and child live together under a certain level of public pensions, the child live in the location with potential for highest earnings, where the parents never move in the second period, if the level of public pensions rises and reaches a threshold value.

One possible extension of this model is to incorporate families with two or more children, while the percentage of families with an only child, as considered in the present study, has been increasing in fertility-declining countries such as Japan. Konrad *et al*. (2002) and Rainer and Siedler (2009) argue that the presence of a sibling crucially affects the residential choice of children. In Konrad *et al*., while only children live with their parents, older children with a sibling may move further away from their parents to induce younger children to live with their parents in order to take care of them. Rainer and Siedler show that children with a sibling are likely to live further away from their parents than only children, while the birth order does not fully explain siblings' location unlike Konrad *et al*. Therefore, it would be worth examining the effects of public pensions on the location choice of children with a sibling (or siblings) for comparison with the results of the present study so as to round out the analysis on public pensions and geographic mobility in the family.

Appendix

Derivation of (12)-(14) and (18)

Differentiating (8) and (9) with respect to π, g_k and g_p yields

(A.1) $$\begin{bmatrix} \frac{-1}{C_k^2}+\frac{-\rho}{C_p^2}, & \frac{-1}{C_k^2} \\ \frac{-1}{C_k^2}, & \frac{-1}{C_k^2}+\frac{-\alpha}{G_k^2}+\frac{-\rho\alpha\gamma^2}{G_p^2} \end{bmatrix} \begin{bmatrix} d\pi \\ dg_k \end{bmatrix} = \begin{bmatrix} \frac{-\rho}{C_p^2} \\ \frac{\gamma\alpha}{G_k^2}+\frac{\rho\alpha\gamma}{G_p^2} \end{bmatrix}.$$

From (A.1), we have

$$\frac{\partial\pi^+}{\partial g_p} = \frac{\begin{vmatrix} \frac{-\rho}{C_p^2}, & \frac{-1}{C_k^2} \\ \frac{\gamma\alpha}{G_k^2}+\frac{\rho\alpha\gamma}{G_p^2}, & \frac{-1}{C_k^2}+\frac{-\alpha}{G_k^2}+\frac{-\rho\alpha\gamma^2}{G_p^2} \end{vmatrix}}{D}.$$

Adding the second column to the first column of the determinant yields

$$\frac{\partial\pi^+}{\partial g_p} = \frac{\begin{vmatrix} \frac{-1}{C_k^2}+\frac{-\rho}{C_p^2}, & \frac{-1}{C_k^2} \\ \frac{-1}{C_k^2}+\alpha(1-\gamma)\left(\frac{-1}{G_k^2}+\frac{\rho\gamma}{G_p^2}\right), & \frac{-1}{C_k^2}+\frac{-\alpha}{G_k^2}+\frac{-\rho\alpha\gamma^2}{G_p^2} \end{vmatrix}}{D}$$

$$=1+\frac{\alpha(1-\gamma)}{D}\frac{1}{C_k^2}\left(\frac{-1}{G_k^2}+\frac{\rho\gamma}{G_p^2}\right)>0.$$

Thus, we obtain (12). Similarly, from (A.1) we have

$$\frac{\partial g_k^+}{\partial g_p} = \frac{\begin{vmatrix} \frac{-1}{C_k^2}+\frac{-\rho}{C_p^2}, & \frac{-\rho}{C_p^2} \\ \frac{-1}{C_k^2}, & \frac{\alpha\gamma}{G_k^2}+\frac{\rho\alpha\gamma}{G_p^2} \end{vmatrix}}{D}.$$

Multiplying the second column of the determinant by (-1) and adding the first column to the second column yields

$$\frac{\partial g_k^+}{\partial g_p} = \frac{-\begin{vmatrix} \frac{-1}{C_k^2}+\frac{-\rho}{C_p^2}, & \frac{-1}{C_k^2} \\ \frac{-1}{C_k^2}, & \frac{-1}{C_k^2}-\frac{\alpha\gamma}{G_k^2}-\frac{\rho\alpha\gamma}{G_p^2} \end{vmatrix}}{D}$$

$$= \frac{-\begin{vmatrix} \frac{-1}{C_k^2}+\frac{-\rho}{C_p^2}, & \frac{-1}{C_k^2} \\ \frac{-1}{C_k^2}, & \frac{-1}{C_k^2}+\frac{-\alpha}{G_k^2}+\frac{-\rho\alpha\gamma^2}{G_p^2}+\alpha(1-\gamma)\left(\frac{1}{G_k^2}+\frac{-\rho\gamma}{G_p^2}\right) \end{vmatrix}}{D}$$

$$= -1+\frac{\alpha(1-\gamma)}{D}\left(\frac{1}{C_k^2}+\frac{\rho}{C_p^2}\right)\left(\frac{1}{G_k^2}-\frac{\rho\gamma}{G_p^2}\right) < 0.$$

Thus, we obtain (13).

When $\pi = 0$, differentiating (9) with respect to g_k and g_p yields

$$\frac{-1}{C_k^2}dg_k + \frac{-\alpha}{G_k^2}(dg_k+\gamma dg_p) + \frac{-\rho\alpha\gamma}{G_p^2}(dg_p+\gamma dg_k) = 0.$$

Thus, we obtain (14). When $\gamma = 1$, we have $G = G_K = G_p = g_k + g_p$, and (14) is rewritten as follows:

(A.2)
$$\frac{\partial g_k^0}{\partial g_p} = \frac{-\alpha(1+\rho)/G^2}{(1/C_k^2)+[\alpha(1+\rho)/G^2]}.$$

From (9) with $\gamma = 1$, we have $1/C_k = \alpha(1+\rho)/G$. Substituting this equation into (A.2) yields (18).

The sign of $d\tilde{U}_p/dp$ for $k-\overline{\delta} < p \le \tilde{p}$

(i) $\hat{p} < p \le \tilde{p}$

Differentiating $\tilde{U}_p$ with $\pi = 0$ with respect to p and using the envelop theorem yield

(A.3)

$$\frac{d\tilde{U}_p}{dp} = -\gamma'(\delta)\left\{-\frac{1}{Y_p - g_p + T}\frac{\partial g_p}{\partial \gamma} + \frac{\alpha}{g_p + \gamma(\delta) g_k}\left(\frac{\partial g_p}{\partial \gamma} + g_k + \gamma(\delta)\frac{\partial g_k}{\partial \gamma} + \gamma(\delta)\frac{\partial g_k}{\partial g_p}\frac{\partial g_p}{\partial \gamma}\right)\right\}$$

$$= -\gamma'(\delta)\left\{\frac{\alpha}{g_p + \gamma(\delta) g_k}\left(g_k + \gamma(\delta)\frac{\partial g_k}{\partial \gamma}\right)\right\}.$$

Furthermore, differentiating (9) with respect to g_k and γ in order to derive the sign of $\partial g_k / \partial \gamma$ in (A.3) yields

$$-\left(\frac{1}{C_k^2} + \frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2}\right) dg_k + \left(\frac{\rho\alpha}{G_p} - \frac{\alpha g_p}{G_k{}^2} - \frac{\rho\alpha\gamma g_k}{G_p{}^2}\right) d\gamma = 0 .$$

Hence we have

$$\text{(A.4)} \qquad \frac{\partial g_k}{\partial \gamma} = \left(\frac{\rho\alpha}{G_p} - \frac{\alpha g_p}{G_k{}^2} - \frac{\rho\alpha\gamma g_k}{G_p{}^2}\right) \Big/ \left(\frac{1}{C_k^2} + \frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2}\right).$$

The sign of $(\rho\alpha / G_p) - (\alpha g_p / G_k{}^2) - (\rho\alpha\gamma g_k / G_p{}^2)$ in the right-hand side of (A.4) is indeterminate and so is the sign of $d\tilde{U}_p / dp$ from (A.3).

We could interpret $(\rho\alpha / G_p) - (\alpha g_p / G_k{}^2) - (\rho\alpha\gamma g_k / G_p{}^2)$ in (A.4) as follows. It is the marginal change of the marginal benefit of g_k for the child, $(\alpha / G_k) + (\rho\alpha\gamma(\delta) / G_p)$, through an increase in $\gamma(\delta)$. A rise in $\gamma(\delta)$ increases the marginal benefit of g_k directly because it increases G_p by one unit, given g_k. On the other hand, a rise in $\gamma(\delta)$ decreases the marginal benefit of g_k indirectly, because it increases G_k and G_p and thus decreases the marginal utility of G_k and G_p. If the former direct effect dominates the latter indirect effect through the increase in G_k and G_p, we obtain $\partial g_k / \partial \gamma > 0$, which implies $d\tilde{U}_p / dp > 0$.

(ii) $k - \overline{\delta} < p \le \hat{p}$

Differentiating $\tilde{U}_p$ with respect to p and using the envelop theorem yield

(A.5)

$$\frac{d\tilde{U}_p}{dp} = -\gamma'(\delta)\left\{\frac{1}{Y_p - g_p + T}\left(\frac{\partial \pi}{\partial \gamma} + \frac{\partial \pi}{\partial g_p}\frac{\partial g_p}{\partial \gamma} - \frac{\partial g_p}{\partial \gamma}\right)\right.$$

$$\left. + \frac{\alpha}{g_p + \gamma(\delta) g_k}\left(\frac{\partial g_p}{\partial \gamma} + g_k + \gamma(\delta)\frac{\partial g_k}{\partial \gamma} + \gamma(\delta)\frac{\partial g_k}{\partial g_p}\frac{\partial g_p}{\partial \gamma}\right)\right\}$$

$$= -\gamma'(\delta)\left\{\frac{1}{Y_p - g_p + T}\frac{\partial \pi}{\partial \gamma} + \frac{\alpha}{g_p + \gamma(\delta) g_k}\left(g_k + \gamma(\delta)\frac{\partial g_k}{\partial \gamma}\right)\right\}.$$

Furthermore, differentiating (8) and (9) with respect to π, g_k and γ in order to derive the sign of $\partial \pi / \partial \gamma$ and $\partial g_k / \partial \gamma$ in (A.5) yields

(A.6) $$\begin{bmatrix} \frac{-1}{C_k^2} + \frac{-\rho}{C_p^2}, & \frac{-1}{C_k^2} \\ \frac{-1}{C_k^2}, & \frac{-1}{C_k^2} + \frac{-\alpha}{G_k^2} + \frac{-\rho\alpha\gamma^2}{G_p^2} \end{bmatrix} \begin{bmatrix} d\pi \\ dg_k \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\alpha g_p}{G_k^2} + \frac{\rho\alpha\gamma g_k}{G_p^2} - \frac{\rho\alpha}{G_p} \end{bmatrix} d\gamma.$$

From (A.6), we have

(A.7) $$\frac{\partial \pi}{\partial \gamma} = \frac{\begin{vmatrix} 0, & \frac{-1}{C_k^2} \\ \frac{\alpha g_p}{G_k^2} + \frac{\rho\alpha\gamma g_k}{G_p^2} - \frac{\rho\alpha}{G_p}, & \frac{-1}{C_k^2} + \frac{-\alpha}{G_k^2} + \frac{-\rho\alpha\gamma^2}{G_p^2} \end{vmatrix}}{D}$$

$$= \frac{1}{C_k^2}\left(\frac{\alpha g_p}{G_k^2} + \frac{\rho\alpha\gamma g_k}{G_p^2} - \frac{\rho\alpha}{G_p}\right) / D.$$

Similarly, from (A.6) we have

(A.8) $$\frac{\partial g_k}{\partial \gamma} = \frac{\begin{vmatrix} \frac{-1}{C_k^2} + \frac{-\rho}{C_p^2}, & 0 \\ \frac{-1}{C_k^2}, & \frac{\alpha g_p}{G_k^2} + \frac{\rho\alpha\gamma g_k}{G_p^2} - \frac{\rho\alpha}{G_p} \end{vmatrix}}{D}$$

$$= -\left(\frac{1}{C_k^2} + \frac{\rho}{C_p^2}\right)\left(\frac{\alpha g_p}{G_k^2} + \frac{\rho\alpha\gamma g_k}{G_p^2} - \frac{\rho\alpha}{G_p}\right) / D.$$

These signs are indeterminate because the sign of $(\rho\alpha / G_p) - (\alpha g_p / G_k{}^2) - (\rho\alpha\gamma g_k / G_p{}^2)$ is indeterminate and then so is the sign of

$d\tilde{U}_p / dp$ from (A.5).

Sufficient condition for $d^2\tilde{U}_p / (dp)^2 \geq 0$ **(**$\tilde{U}_p$ **is convex in** p**) given** k

We have

$$\tilde{U}_p = \log(Y_p - g_p + \pi + T) + \alpha \log(g_p + \gamma(\delta) g_k),$$

where $\delta = k - p$. The first-order differentiation of $\tilde{U}_p$ with respect to p is

$$\frac{d\tilde{U}_p}{dp} = -\frac{d\tilde{U}_p}{d\gamma}\gamma'(\delta). \tag{A.9}$$

The second-order differentiation of $\tilde{U}_p$ with respect to p is

$$\begin{aligned} \frac{d^2\tilde{U}_p}{dp^2} &= \frac{d^2\tilde{U}_p}{d\gamma^2}(\gamma'(\delta))^2 + \frac{d\tilde{U}_p}{d\gamma}\gamma''(\delta) \\ &= \frac{d\tilde{U}_p}{d\gamma}\frac{(\gamma')^2}{\gamma}\left[\eta_{\gamma'(\delta)} / \eta_{\gamma(\delta)} - \eta_{\tilde{U}'_p(\gamma)}\right], \end{aligned} \tag{A.10}$$

where

$$\eta_{\tilde{U}'_p(\gamma)} \equiv -\frac{d^2\tilde{U}_p}{d\gamma^2}\frac{\gamma}{d\tilde{U}_p / d\gamma},$$

$$\eta_{\gamma(\delta)} \equiv -\frac{\gamma'(\delta)}{\gamma(\delta)}\delta \ (>0),$$

$$\eta_{\gamma'(\delta)} \equiv -\frac{\gamma''(\delta)}{\gamma'(\delta)}\delta \ (>0).$$

As shown above, the sign of $d\tilde{U}_p / dp$ is generally indeterminate, but we have $d\tilde{U}_p / dp \geq 0$ if $(\rho\alpha / G_p) - (\alpha g_p / G_k^2) - (\rho\alpha\gamma g_k / G_p^2) > 0$. Under this assumption, we also have $d\tilde{U}_p / d\gamma \geq 0$ from (A.9). From (A.10), therefore, we derive the following proposition.

Proposition A1. If $(\rho\alpha / G_p) - (\alpha g_p / G_k^2) - (\rho\alpha\gamma g_k / G_p^2) > 0$ and $\eta_{\gamma'(\delta)} / \eta_{\gamma(\delta)} \geq \eta_{\tilde{U}'_p(\gamma)}$, then we have $d^2\tilde{U}_p / (dp)^2 \geq 0$ **(**$\tilde{U}_p$ is convex in p).

We now provide an interpretation of Proposition A1. $\eta_{\gamma'(\delta)} / \eta_{\gamma(\delta)} \geq \eta_{\tilde{U}'_p(\gamma)}$ in Proposition A1 is rewritten as $\eta_{\gamma'(\delta)} \geq \eta_{\tilde{U}'_p(\gamma)} \cdot \eta_{\gamma(\delta)}$. From (A.9), $d^2\tilde{U}_p / (dp)^2$ (the effect of p on $d\tilde{U}_p / dp$) can be decomposed into the effect of p on $d\tilde{U}_p / d\gamma(\delta)$ and the effect on $\gamma'(\delta)$. The effect of δ (or p) on $d\tilde{U}_p / d\gamma(\delta)$ is expressed as $\eta_{\tilde{U}'_p(\gamma)} \cdot \eta_{\gamma(\delta)}$, which is composed of $\eta_{\tilde{U}'_p(\gamma)}$ (the elasticity of $d\tilde{U}_p / d\gamma$ to γ) and $\eta_{\gamma(\delta)}$ (the elasticity of $\gamma(\delta)$ to δ). A 1 percent increase in δ (a 1 percent decrease in p) results in a $\eta_{\tilde{U}'_p(\gamma)} \cdot \eta_{\gamma(\delta)}$ percent change in $d\tilde{U}_p / d\gamma(\delta)$. On the other hand, the effect of δ on $\gamma'(\delta)$ is expressed as $\eta_{\gamma'(\delta)}$, and a 1 percent increase in δ results in a $\eta_{\gamma'(\delta)}$ percent increase in $\gamma'(\delta)$ because $d^2\gamma(\delta) / d\delta^2 > 0$. When the latter effect dominates the former one, a 1 percent increase in δ results in a $(\eta_{\gamma'(\delta)} - \eta_{\tilde{U}'_p(\gamma)} \cdot \eta_{\gamma(\delta)})$ percent increase in $d\tilde{U}_p / dp$, which implies that $\tilde{U}_p$ is convex in δ (or p).

The above discussion implies that whether $\eta_{\gamma'(\delta)} / \eta_{\gamma(\delta)} \geq \eta_{\tilde{U}'_p(\gamma)}$ holds or not depends on the shape of $\gamma(\delta)$. Particularly for $\tilde{p} < p \leq k$, we can derive the convexity of $\tilde{U}_p$ in p only from the condition on the shape of $\gamma(\delta)$. In this region, from $g_p = \pi = 0$, we have

$$\tilde{U}_p = \log(Y_p + T) + \alpha \log(\gamma(\delta) g_k).$$

Second-order differentiation of $\tilde{U}_p$ with respect to p yields

$$\text{(A.11)} \qquad \begin{aligned} \frac{d^2\tilde{U}_p}{(dp)^2} &= \frac{\alpha[\gamma''(\delta)\gamma(\delta) - (\gamma'(\delta))^2]}{(\gamma(\delta))^2} \\ &= \frac{\alpha(\gamma'(\delta))^2}{(\gamma(\delta))^2}\left[\eta_{\gamma'(\delta)} / \eta_{\gamma(\delta)} - 1\right]. \end{aligned}$$

From (A.11), noting that $\eta_{\tilde{U}'_p(\gamma)} = 1$ in this region,[19] we have that, if

---

[19] When $g_p = \pi = 0$, we have $\tilde{U}_p = \log(Y_p + T) + \alpha \log \gamma(\delta) g_k$. Since we have $d\tilde{U}_p / d\gamma = \alpha / \gamma$ and $d^2\tilde{U}_p / d\gamma^2 = -\alpha / \gamma^2$, we obtain

$$\eta_{\tilde{U}'_p(\gamma)} \equiv -\frac{d^2\tilde{U}_p}{d\gamma^2} \frac{\gamma}{d\tilde{U}_p / d\gamma} = \frac{\alpha}{\gamma^2} \frac{\gamma^2}{\alpha} = 1.$$

$\eta_{\gamma'(\delta)} / \eta_{\gamma(\delta)} > 1\ (= \eta_{\tilde{U}'_p(\gamma)})$, then $\tilde{U}_p$ is convex in p.

Sufficient conditions for the concavity of $\tilde{U}_p^D (\equiv \tilde{U}_p\big|_{p=k} - \tilde{U}_p\big|_{p=0})$ in k

Noting that we have $g_k\big|_{p=k} = \theta(Y_k(k) - T)$ $(\theta \equiv [\alpha(1+\rho)]/[1+\alpha(1+\rho)])$ when the parents move to the child's location and they live in the same home ($p = k$), we have

$$
\begin{aligned}
\tilde{U}_p\big|_{p=k} &= \log(Y_p + T) + \alpha \log(g_k\big|_{p=k}) \\
&= \log(Y_p + T) + \alpha \log(\theta(Y_k(k) - T)).
\end{aligned} \tag{A.12}
$$

Differentiating (A.12) with respect to k yields

$$
\frac{d\tilde{U}_p\big|_{p=k}}{dk} = \frac{\alpha Y_k'(k)}{Y_k(k) - T}. \tag{A.13}
$$

On the other hand, when the parents do not move and remain in their present location ($p = 0$) and the child lives at k, we have

$$
\tilde{U}_p\big|_{p=0} = \log(Y_p - g_p\big|_{p=0} + \pi + T) + \alpha \log(g_p\big|_{p=0} + \gamma(k) g_k\big|_{p=0}). \tag{A.14}
$$

Differentiating (A.14) with respect to k and using the envelop theorem for $g_p\big|_{p=0}$ yields

$$
\frac{d\tilde{U}_p\big|_{p=0}}{dk} = \frac{(\partial\pi / \partial Y_k(k)) Y_k'(k)}{C_p\big|_{p=0}} + \frac{\alpha[\gamma(\partial g_k\big|_{p=0} / \partial Y_k(k)) Y_k'(k) + \gamma'(k) g_k\big|_{p=0}]}{G_p\big|_{p=0}}. \tag{A.15}
$$

Subtracting (A.15) from (A.13) yields

$$
\begin{aligned}
\frac{d\tilde{U}_p^D}{dk} &\equiv \frac{d\tilde{U}_p}{dk}\Big|_{p=k} - \frac{d\tilde{U}_p}{dk}\Big|_{p=0} \\
&= \frac{-(\partial\pi / \partial Y_k) Y_k'(k)}{C_p\big|_{p=0}} + \frac{\alpha Y_k'(k)}{Y_k - T} - \frac{\alpha\gamma(\partial g_k / \partial Y_k) Y_k'(k) + \alpha\gamma'(k) g_k\big|_{p=0}}{G_p\big|_{p=0}}.
\end{aligned} \tag{A.16}
$$

Differentiating (A.16) with respect to k, we have

$$\text{(A.17)} \quad \begin{aligned}\frac{d^2\tilde{U}_p^D}{dk^2} &= \left(\frac{(\partial\pi / \partial Y_k)^2}{(C_p\big|_{p=0})^2} + \frac{\alpha\gamma^2 (\partial g_k\big|_{p=0} / \partial Y_k)^2}{(G_p\big|_{p=0})^2} - \frac{\alpha}{(Y_k - T)^2} \right)(Y_k'(k))^2 \\ &\quad - \left(\frac{\partial^2\pi / \partial Y_k^2}{C_p\big|_{p=0}} + \frac{\alpha\gamma(\partial^2 g_k\big|_{p=0} / \partial Y_k^2)}{G_p\big|_{p=0}} \right) Y_k'(k) \\ &\quad + \left(\frac{\alpha(\gamma'(k))^2 (g_k\big|_{p=0})^2}{(G_p\big|_{p=0})^2} - \frac{\alpha\gamma''(k) g_k\big|_{p=0}}{G_p\big|_{p=0}} \right).\end{aligned}$$

The first term in (A.17) is indeterminate, because the first and second terms in the parenthesis are positive while the third term in the parenthesis is negative.

We next examine the second term in (A.17). To do this, we examine the effects of Y_k on π and g_k. We have

$$\text{(A.18)} \quad \frac{\partial\pi}{\partial Y_k} = \frac{1}{D}\left[\frac{1}{C_k^2}\left(\frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2} \right) \right] > 0 \ \left(0 < \frac{\partial\pi}{\partial Y_k} < 1 \right),$$

$$\text{(A.19)} \quad \frac{\partial g_k\big|_{p=0}}{\partial Y_k} = \frac{1}{D}\left[\frac{1}{C_k^2}\frac{\rho}{C_p^2} \right] > 0 \ \left(0 < \frac{\partial g_k\big|_{p=0}}{\partial Y_k} < 1 \right),$$

where

$$\text{(A.20)} \quad \begin{aligned} D &= \frac{1}{C_k^2}\left(\frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2} \right) + \frac{\rho}{C_p^2}\left(\frac{1}{C_k^2} + \frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2} \right) \\ &= \left(1 + \frac{1}{\rho} \right)\frac{1}{C_k^2}\left(\frac{\alpha}{G_k^2} + \frac{\rho\alpha\gamma^2}{G_p^2} \right) + \left(\frac{1}{C_k^2}\frac{\rho}{C_p^2} \right) > 0. \end{aligned}$$

Substituting (A.18) and (A.19) into (A.20) and dividing both hand sides of the resulting equation by D yields

$$\text{(A.21)} \quad \left(1 + \frac{1}{\rho} \right)\frac{\partial\pi}{\partial Y_k} + \frac{\partial g_k\big|_{p=0}}{\partial Y_k} = 1.$$

Furthermore, differentiating (A.21) with respect to Y_k yields

$$\text{(A.22)} \quad \left(1 + \frac{1}{\rho} \right)\frac{\partial^2\pi}{\partial Y_k^2} + \frac{\partial^2 g_k\big|_{p=0}}{\partial Y_k^2} = 0.$$

From (A.22), it turns out that the sign of $\partial^2\pi / \partial Y_k^2$ is opposite to that of $\partial^2 g_k\big|_{p=0} / \partial Y_k^2$. Substituting (A.22) into the second term in (A.17) yields

$$\text{(A.23)}\ -\left(\frac{\partial^2\pi/\partial Y_k^2}{C_p\big|_{p=0}}+\frac{\alpha\gamma(\partial^2 g_k\big|_{p=0}/\partial Y_k^2)}{G_p\big|_{p=0}}\right)Y_k'(k)=-\left\{\left(\frac{\partial^2\pi}{\partial Y_k^2}\right)\frac{[\rho G_p-(1+\rho)\alpha\gamma C_p]}{\rho C_p\big|_{p=0}G_p\big|_{p=0}}\right\}Y_k'(k)\ .$$

From (A.23), if $\partial^2\pi/\partial Y_k^2 \geq 0\ (\partial^2 g_k/\partial Y_k^2 \leq 0)$, the second term in (A.17) is non-positive because of $\rho G_p-(1+\rho)\alpha\gamma C_p>0$.[20]

Finally, the third term in (A.17) can be rewritten as follows.

$$\text{(A.24)}\ \frac{\alpha(\gamma'(k))^2(g_k\big|_{p=0})^2}{(G_p\big|_{p=0})^2}-\frac{\alpha\gamma''(k)g_k\big|_{p=0}}{G_p\big|_{p=0}}=-\frac{\alpha g_k\big|_{p=0}}{G_p\big|_{p=0}}\frac{(\gamma')^2}{\gamma}\left[\eta_{\gamma'(\delta)}/\eta_{\gamma(\delta)}-\eta_{\tilde{U}_p'^D(\gamma)}\right].$$

From (A.24), if the sufficient condition of Proposition A1 holds, the third term in (A.17) is negative.

As shown above, the sign of (A.17) could be negative, but generally it is ambiguous. However, we can obtain clearer results for the following regions of k.

(i) $0 \leq k \leq \tilde{k}$

In this region of k, from $g_p=\pi=0$, we have

$$\text{(A.25)}\qquad \tilde{U}_p\big|_{p=0}=\log(Y_p+T)+\alpha\log(\gamma(k)g_k\big|_{p=0})\ .$$

From (9), we have

$$\text{(A.26)}\qquad g_k\big|_{p=0}=\theta(Y_k(k)-T)\ \left(\theta\equiv\left[\alpha(1+\rho)\right]/\left[1+\alpha(1+\rho)\right]\right).$$

Substituting (A.26) into (A.25) and differentiating the resulting equation with respect to

---

[20] From (15) with equality, we have $C_p/G_p=(1-\pi_1)/\alpha(1+\gamma g_{k1})$, where $\pi_1\equiv\partial\pi^+/\partial g_p$ and $g_{k1}\equiv\partial g_k^+/\partial g_p$. Substituting $C_p/G_p=(1-\pi_1)/\alpha(1+\gamma g_{k1})$ into $\rho G_p-(1+\rho)\alpha\gamma C_p$ yields

$$\begin{aligned}\rho G_p-(1+\rho)\alpha\gamma C_p &= G_p\left[\rho-(1+\rho)\alpha\gamma\frac{C_p}{G_p}\right]\\ &= G_p\left[\rho-(1+\rho)\alpha\gamma\frac{(1-\pi_1)\gamma}{1+\gamma g_{k1}}\right]=\frac{G_p}{1+\gamma g_{k1}}\left[\rho(1+\gamma g_{k1})-(1+\rho)(1-\pi_1)\gamma\right]\\ &= \frac{G_p}{1+\gamma g_{k1}}\left[\rho(1-\gamma)+\rho\gamma(\pi_1+g_{k1})+(\pi_1-1)\gamma\right].\end{aligned}$$

From (12) and (13), we have $\rho\gamma(\pi_1+g_{k1})+(\pi_1-1)\gamma=0$, which implies that

$$\rho G_p-(1+\rho)\alpha\gamma C_p=\frac{G_p}{1+\gamma g_{k1}}\rho(1-\gamma)>0\ .$$

k yields

(A.27) $$\frac{d\tilde{U}_p\big|_{p=0}}{dk}=\frac{\alpha[\gamma'(k)(Y_k-T)+\gamma(k)Y_k'(k)]}{\gamma(k)(Y_k-T)}.$$

Subtracting (A.27) from (A.13) yields

(A.28) $$\begin{aligned}\frac{d\tilde{U}_p^D}{dk}&=\frac{\alpha Y_k'(k)}{Y_k(k)-T}-\frac{\alpha[\gamma'(k)(Y_k-T)+\gamma(k)Y_k'(k)]}{\gamma(k)(Y_k-T)}\\&=-\frac{\alpha\gamma'(k)}{\gamma(k)}.\end{aligned}$$

Furthermore, differentiating (A.28) with respect to k, we have

(A.29) $$\begin{aligned}\frac{d^2\tilde{U}_p^D}{dk^2}&=-\frac{\alpha[\gamma''(k)\gamma(k)-(\gamma'(k))^2]}{(\gamma(k))^2}\\&=\frac{-\alpha(\gamma'(k))^2[\eta_{\gamma'}/\eta_\gamma-1]}{(\gamma(k))^2}.\end{aligned}$$

This is also obtained by substituting $g_p=\pi=0$ and $g_k\big|_{p=0}=\theta(Y_k(k)-T)$ into (A.17).

From (A.29), we obtain the following proposition.

Proposition A2.
For $0\le k\le\tilde{k}$, if we have $\eta_{\gamma'}/\eta_\gamma>1$ (the sufficient condition of Proposition A1 is satisfied), $\tilde{U}_p^D$ is concave in k.

(ii) $k\ge\overline{\delta}$

In this region of k, from $\gamma=0$, we have

(A.30) $$\tilde{U}_p\big|_{p=0}=\log(Y_p-g_p\big|_{p=0}+\pi+T)+\alpha\log(g_p\big|_{p=0}).$$

Differentiating (A.30) with respect to k and using the envelope theorem for $g_p\big|_{p=0}$ yields

(A.31) $$\frac{d\tilde{U}_p\big|_{p=0}}{dk}=\frac{(\partial\pi/\partial Y_k(k))Y_k'(k)}{C_p\big|_{p=0}},$$

where

(A.32) $$\frac{\partial \pi}{\partial Y_k} = \frac{1}{D}\frac{1}{C_k^2}\frac{\alpha}{G_k^2} = \frac{1}{D}\frac{1}{C_k^2}\frac{1}{\alpha C_k^2}.$$

Substituting (8) with $\gamma = 0$ ($C_p = \rho C_k$) and (9) with $\gamma = 0$ ($G_k = \rho C_k$) into (A.20) yields $1/D = \rho\alpha C_k^4/(\rho+\alpha+1)$. Substituting this equation into (A.32) yields

(A.33) $$\frac{\partial \pi}{\partial Y_k} = \frac{\rho}{\rho+\alpha+1}.$$

Hence, we have $d\tilde{U}_p\big|_{p=0}/dk > 0$.

Subtracting (A.31) from (A.13) yields

(A.34) $$\frac{d\tilde{U}_p^D}{dk} = \frac{-(\partial\pi/\partial Y_k)Y_k'(k)}{C_p\big|_{p=0}} + \frac{\alpha Y_k'(k)}{Y_k(k)-T}.$$

This is also obtained by substituting $\gamma = 0$ into (A.16). Furthermore, differentiating (A.34) with respect to k, we have

(A.35) $$\frac{d^2\tilde{U}_p^D}{dk^2} = \left[\left(\frac{(\partial\pi/\partial Y_k)^2}{(C_p\big|_{p=0})^2} - \frac{\alpha}{(Y_k-T)^2}\right)(Y_k'(k))^2 - \left(\frac{\partial^2\pi/\partial Y_k^2}{(C_p\big|_{p=0})^2}\right)Y_k'(k)\right].$$

This is also obtained by substituting $\gamma = 0$ into (A.17). From (A.33) we have

(A.36) $$\frac{\partial^2\pi}{\partial Y_k^2} = 0.$$

Substituting (A.33), (A.36) and (9) with $\gamma = 0$ into (A.35) yields

(A.37) $$\frac{d^2\tilde{U}_p^D}{dk^2} = \left[\frac{\alpha}{(g_k\big|_{p=0})^2(\rho+\alpha+1)^2} - \frac{1}{(Y_k-T)^2}\right]\alpha(Y_k'(k))^2.$$

$g_k\big|_{p=0}$ is obtained by solving (8) with equality, (9) and (15) with equality and $\gamma = 0$ as a simultaneous equation system. Noting that $\partial\pi/\partial g_p = (1+\alpha)/(\rho+\alpha+1)$,[21] we

[21] This is obtained by substituting $\gamma = 0$, (9) with $\gamma = 0$ and $1/D = \rho\alpha C_k^4/(\rho+\alpha+1)$ into (12):

have

(A.38) $$g_k\Big|_{p=0} = \frac{\alpha[\alpha^2 + ((Y_k - T) + (Y_p + T))\rho + \alpha(1+\rho)]}{\rho(1+\alpha+\rho)}.$$

Substituting (A.38) into (A.37) yields

(A.39) $$\begin{aligned}\frac{d^2\tilde{U}_p^D}{dk^2} &= \left[\frac{\rho^2}{\alpha[\alpha^2 + ((Y_k - T) + (Y_p + T))\rho + \alpha(1+\rho)]^2} - \frac{1}{(Y_k - T)^2}\right]\alpha(Y_k'(k))^2 \\ &= \left[\frac{\rho^2(Y_k - T)^2(1-\alpha) - Z}{\alpha[\alpha^2 + ((Y_k - T) + (Y_p + T))\rho + \alpha(1+\rho)]^2(Y_k - T)^2}\right]\alpha(Y_k'(k))^2,\end{aligned}$$

where

$$\begin{aligned}Z &= \alpha[\alpha^2 + \alpha(1+\rho)]^2 + \rho^2(Y_p + T)^2 \\ &\quad + 2[\rho^2(Y_k - T)(Y_p + T) + ((Y_k - T) + (Y_p + T))(\rho\alpha^2 + (1+\rho)\rho\alpha)].\end{aligned}$$

Since $Z > 0$, the following proposition is obtained from (A.39).

Proposition A3.

For $k \geq \overline{\delta}$, if we have $\alpha \geq 1$, $\tilde{U}_p^D$ is concave in k.

$$\begin{aligned}\frac{\partial \pi}{\partial g_p} &= 1 - \left(\frac{1}{D}\frac{\alpha}{C_k^2}\frac{1}{g_k^2}\right) = 1 - \left(\frac{\rho\alpha C_k^4}{\rho+\alpha+1}\frac{\alpha}{C_k^2}\frac{1}{\alpha^2 C_k^2}\right) \\ &= \frac{1+\alpha}{\rho+\alpha+1}.\end{aligned}$$

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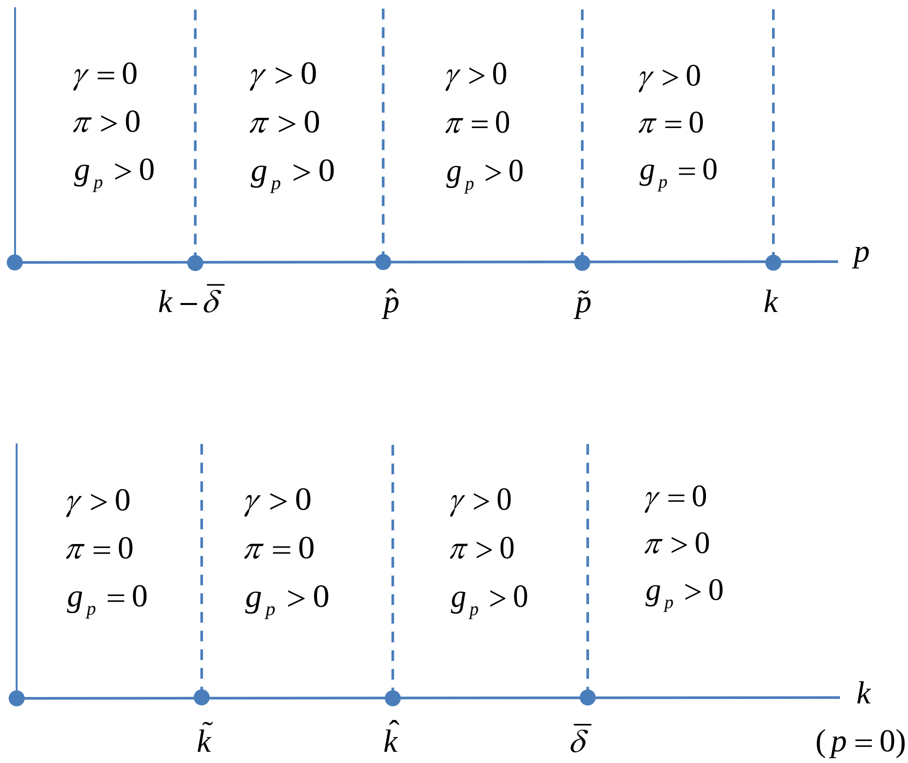

Figure 1: Values of γ, π and g_p in each range of p and k

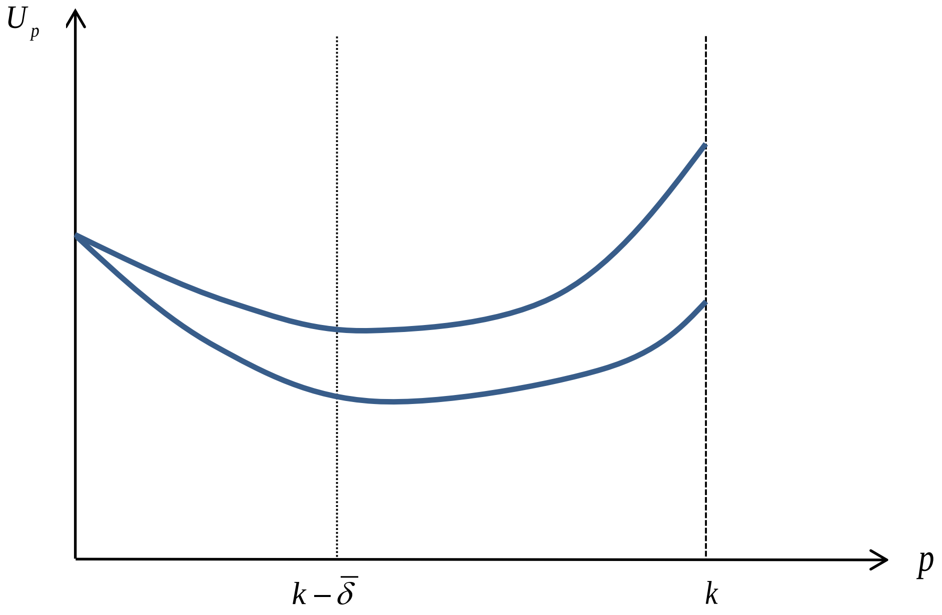

Figure 2: Parental utility function: the upper one corresponds to the case of $p=k$ and the lower one corresponds to the case of $p=0$ in the equilibrium

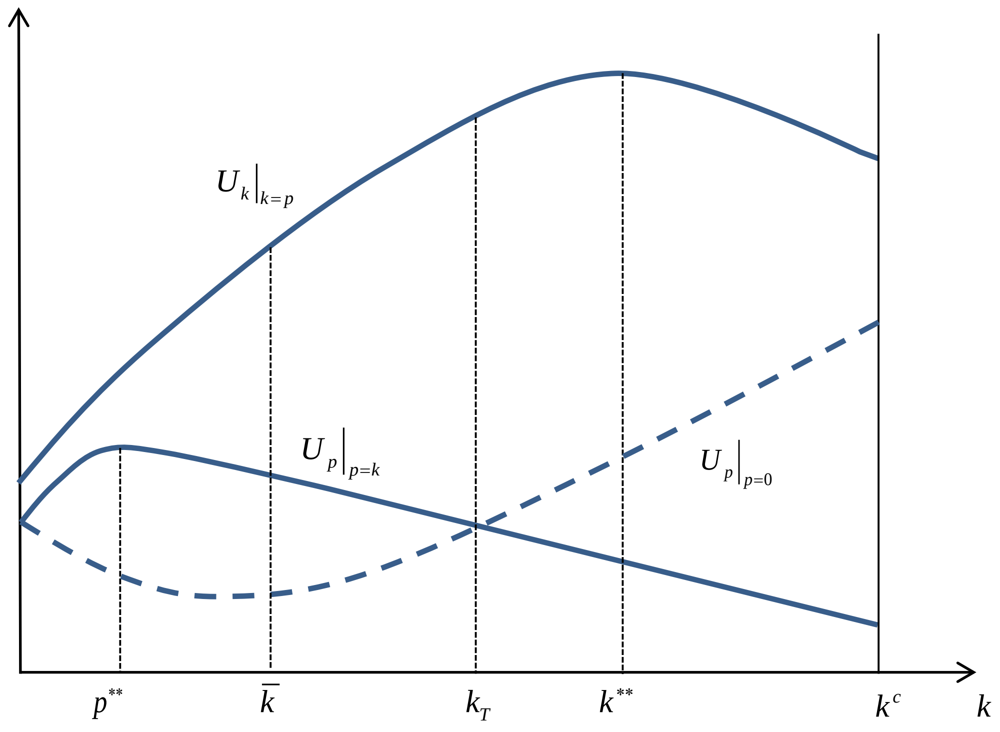

Figure 3: Parental and filial utility as a function of k

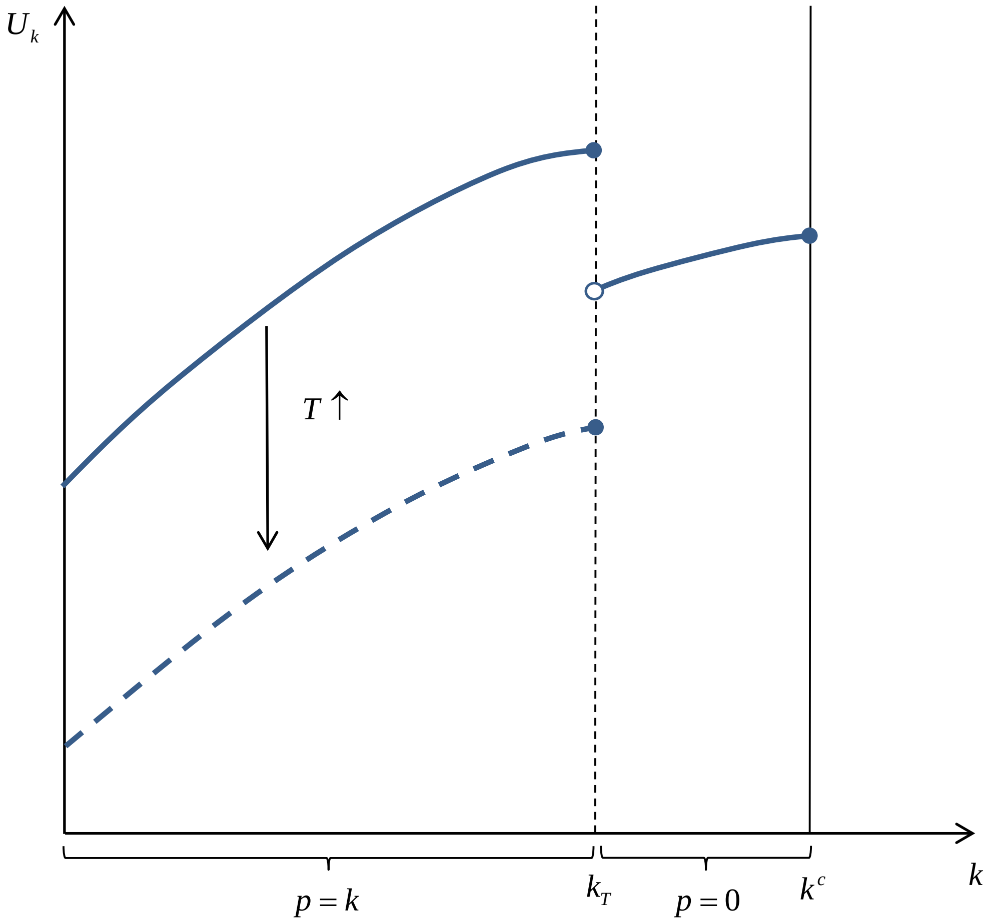

Figure 4: Effect of public pensions on the child's location