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Abstract

This study proposes a dynamic game of two regional monopolies in the same industry. It contains two special features: spillovers and time delays Other than the usual profit maximization through a linear price function in its region, one regional monopoly experiences a cost-increasing spillover associate with the presence of the other regional monopoly. Each regional monopoly needs time to produce output and to acquire information on spillovers. It is demonstrated first that the production delay can be a source of the birth of cyclic oscillations via a Hopf bifurcation. Second, stability loss and gain repeatedly occur according to the relative magnitude between the production and information delays.The number of the regional monopolies is limited to 2 only for analytical simplicity, and will be increased to be more than 3 in a future study.

Keywrods: Regional monopoly, Spillover, Production delay, Information delay, Stability switching curve

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1 Introduction

The primary motivation for this study can be given as follows. Due to high start-up costs and significant economic scales, it is often observed in many countries that industries such as electricity, gas, telecommunications, road, and train networks are mostly monopolized. For example, in Japan, those industries started as public utilities wholly organized by the government; however, they are now divided up and privatized as regional monopolies to prevent high pricing and under-supply. In the electricity industry, for example, the country is divided into nine regions, and one authorized firm exclusively operates in each region. Six regional passenger railroad firms handle all transportation operations concerning the railroad network. The Japan Highway Public Cooperation has been reorganized into several road companies by regions, and the Nippon Telegraph and Telephone Public Cooperation has been split up and privatized. To reorganize existing regional monopolies and to improve market efficiency, deregulation has eliminated barriers to market entry in recent years, and the markets become more "competitive." Nonetheless, the market remains monopolized, with the authorized firm occupying more than 80%. It is then supposed that those regional monopolies supply their products to specific locations due to barriers to competitors and transact only within their regions. The idea we pursue in the study is, contrary to this traditional assumption of no interdependencies, that relations among regional monopolies are not seldom.

The division of one major monopoly creates regional monopolies; thus, their production structures and technologies, working environments, and business strategies are very similar. Although they are still monopolies in their regions even after deregulation, they are also competitors in the same industry. Being interested in technological innovation and the results of R&D(research and development) others undertake and movements of skilled workers, specific know-how, and customer information others possess, some or all of these regional monopolies try to move one step forward to take advantage. Since it is difficult to protect intellectual property, some information on these issues spills over from one firm to another for free. Further, we sometimes hear the following: leakage of top-secrete information, valuable data, unauthorized viewing of customer information, and taking over skilled workers. Some acquire information in a rather aggressive manner without transacting through the markets. Overall, there is a wide variety of spillovers among the regional monopolies; some are beneficial, and others are harmful. This study aims to consider an economic situation in which some regional monopolies in the same industry experience cost-increasing effects due to unfavorable spillovers.

Bischi and Lamantia (2002) analyze how cost-reduction affects existence and uniqueness of Nash equailibrium and study the structure of the basins of two boundedly rational adjustment process in a discrete-time Cournot duopoly framework. Bischi and Tramontana (2009) consider a three-dimensional discret dynamic system to describe the interactions among idustrial clusters. Xiao and Cao (2006) examine stability and Hopf bifurcation in a delayed competitive web sites model, which is based on the Lotka-Volterra competition equations. Their model is three-dimensional and has one delay while our model is two-dimensional but has multiple delays. Matsumoto and Szidarovszky (2023) construct a two-market cobweb model having production delay for agricultural goods and information delay concerning cross-market prices, and show that the unstable steady state bifurcates to various dynamics involving chaotic oscillations. Their model structure is very similar to ours although they consider complementary goods and we assume (perfectly) substitutional goods.

The rest of this paper is organized as follows. Section 2 constructs a basic model of regional monopolies in its first half and obtain stability conditions of the model with production delays and without spillover in the second half. Section 3 considers dynamics with one-way spillover. Section 4 reconstructs the model in which monopolies are symmetric, having the same production delays and the same information delays. Section 5 considers the case in which the sum of the production delays is equal to the sum of the information delays. Lastly Section 6 is devoted to the concluding remarks and the directions of future research.

2 Regional Monopolies with Spillover

Let us now construct a model that involves mutual spillovers. Monopoly firm i produces some amount of good i denoted as q_i and faces a linear price (i.e., an inverse demand) function,

$$p_i = a_i - b_i q_i, \ a_i > 0 \text{ and } b_i > 0. \tag{1}$$

Suppose that monopoly i's cost function is

$$C_i(q_i, q_j) = \left(c_i + \gamma_{ij} q_j\right) q_i.$$

Here c_i is the usual marignal production cost. $\gamma_{ij} q_j$ is the amount of spillover that monopoly i undertakes. It affects the cost of monopoly i related to monopoly j and is assumed to be poportional monopoly j's production level. The marginal production cost with respect to q_i is

$$\frac{\partial C_i}{\partial q_i} = c_i + \gamma_{ij} q_j.$$

The marginal cost effect caused by monopoly j is

$$\frac{\partial C_i}{\partial q_j} = \gamma_{ij} q_i.$$

Through interdependencies, one monopoly receives a variety of favorable and harmful spillovers. Taking all spillovers together, we will have two cases depending on which total effects is dominant. If $\gamma_{ij} > 0$, then monopoly j finally has a spillover associated with a cost-increasing effect on the monopoly i's cost and we call it a adverse spillover effect, and if $\gamma_{ij} < 0$, then monopoly j has a cost-decreasing effect, and we call it a favorable spillover effect. In this study, we focus on the adverse spillover effect.

Assumption 1. $\gamma_{ij} > 0$ and $\gamma_{ji} > 0$: both monopolies receive adverse spillovers increasing production cost.

The profit function is

$$\pi_i(q_i, q_j) = (a_i - b_i q_i)\, q_i - \left(c_i + \gamma_{ij} q_j\right) q_i.$$

The marginal profit is

$$\frac{\partial \pi_i}{\partial q_i} = a_i - c_i - 2b_i q_i - \gamma_{ij} q_j.$$

Then, the optimal output level of regional monopoly i depends on the adverse spillover from firm j (i.e., $\gamma_{ij} q_j > 0$),

$$q_i = r_i^*(q_j) = \frac{1}{2b_i}\left(a_i - c_i - \gamma_{ij} q_j\right).$$

In the same way, the optimal output level of regional monopoly j is

$$q_j = r_j^*(q_i) = \frac{1}{2b_j}\left(a_j - c_j - \gamma_{ji} q_i\right).$$

Solving $q_i = r_i^*(q_j)$ and $q_j = r_j^*(q_i)$ for q_i and q_j simultaneously determines Nash equilibrium outputs,

$$q_i^* = \frac{2b_j d_i - \gamma_{ij} d_j}{4b_i b_j - \gamma_{ij}\gamma_{ji}} \text{ and } q_j^* = \frac{2b_i d_j - \gamma_{ji} d_i}{4b_i b_j - \gamma_{ij}\gamma_{ji}} \tag{2}$$

where we follow the traditional assumption that the maximum price is assumed to be larger than the (pure) marginal cost.

$$d_i = a_i - c_i > 0 \text{ and } d_j = a_j - c_j > 0.$$

To focus on the adverse spillover effects and to have positive equilibrium values, we impose the following:

Assumption 2. $\min[2b_i, 2b_j] > \max\left[\gamma_{ij}, \gamma_{ji}\right]$.

Concerning dynamics of the equilibria, we make two assumptions of the output adjustment. The first one is the gradient method, in which the growth rate of output is adjusted to be proportional to the marginal profit, and the second is that there are production delays on its own production and information delays on the other firm's production. Hence, the dynamic system is

$$\frac{\dot{q}_k(t)}{q_k(t)} = K_k \frac{\partial \pi_k\left(q_k(t-\tau_k), q_{\bar{k}}(t-\tau_{k\bar{k}})\right)}{\partial q_k(t-\tau_k)} \text{ for } K_k > 0,\ k = i, j \text{ and } \bar{k} \neq k$$

or, to be more specific,

$$\begin{aligned} \dot{q}_i(t) &= K_i q_i(t)\left[d_i - 2b_i q_i(t-\tau_i) - \gamma_{ij} q_j(t-\tau_{ij})\right], \\ \dot{q}_j(t) &= K_j q_j(t)\left[d_j - \gamma_{ji} q_i(t-\tau_{ji}) - 2b_j q_j(t-\tau_j)\right]. \end{aligned} \tag{3}$$

Here $\tau_i \geq 0$ and $\tau_j \geq 0$ are production delays and $\tau_{ij} \geq 0$ and $\tau_{ji} \geq 0$ are information delays. It can be checked that $\dot{q}_i(t) = \dot{q}_j(t) = 0$ for $q_k(t-\tau_k) = q_k(t-\tau_{k\bar{k}}) = q_k^*$.

We investigate the local stability of the positive equilibrium point in the linearized version of the dynamic system,

$$\begin{aligned} \dot{q}_i(t) &= -\alpha_i q_i(t-\tau_i) - \beta_i q_j(t-\tau_{ij}), \\ \dot{q}_j(t) &= -\beta_j q_i(t-\tau_{ji}) - \alpha_j q_j(t-\tau_j), \end{aligned} \tag{4}$$

where new parameters are introduced

$$\alpha_k = K_k q_k^* (2b_k) > 0 \text{ and } \beta_k = K_k q_k^* \gamma_{k\bar{k}} > 0 \text{ for } k = i, j \text{ and } \bar{k} \neq k. \tag{5}$$

Under Assumption 2, it is apparent that

$$\alpha_k - \beta_k = K_k q_k^* (2b_k - \gamma_{k\bar{k}}) > 0.$$

The associated characteristic equation is

$$\det \begin{pmatrix} \lambda + \alpha_i e^{-\lambda\tau_i} & \beta_i e^{-\lambda\tau_{ij}} \\ \beta_j e^{-\lambda\tau_{ji}} & \lambda + \alpha_j e^{-\lambda\tau_j} \end{pmatrix} = 0.$$

or the equation is organized as

$$\left(\lambda + \alpha_i e^{-\lambda\tau_i}\right)\left(\lambda + \alpha_j e^{-\lambda\tau_j}\right) - \beta_i \beta_j e^{-\lambda(\tau_{ij}+\tau_{ji})} = 0. \tag{6}$$

As for the benchmark, we ascertain the stability condition in the case of no delays (i.e., $\tau_i = \tau_j = \tau_{i_j} = \tau_{ji} = 0$). The characteristic equation of (6) is reduced to be quadratic in λ,

$$\lambda^2 + (\alpha_i + \alpha_j)\lambda + \left(\alpha_i\alpha_j - \beta_i\beta_j\right) = 0. \tag{7}$$

It is well known that the characteristic roots of (7) have negative real parts if the following two inequalities hold,

$$\alpha_i + \alpha_j > 0$$

and

$$\alpha_i\alpha_j - \beta_i\beta_j = K_i K_j q_i^* q_j^* \left[(2b_i)(2b_j) - \gamma_{ij}\gamma_{ji}\right] > 0.$$

The first inequality always holds, and so does the second due to Assumption 1. Therefore, our first result is summarized as follows:

Theorem 1 *If Assumptions 1 and 2 hold, then the Nash positive equilibrium outputs are locally asymptotically stable when no delays are involved.*

The following point should be noted. The second stability condition always holds whenever either of γ_{ij} or γ_{ji} or both are zero. Hence, we might mention that the spillover effect is a destabilizing factor candidate or a destabilizing ingredient in the sense that a larger $\gamma_{ij}\gamma_{ji}$ might violate the stability condition.

We now detect the effects caused by the production delays on dynamics. To take away the effects by the information delays, we assume no-spillovers (i.e., $\gamma_{ij} = \gamma_{ji} = 0$) with which each regional monopoly behaves as a pure monopoly in its own region. From (2), the optimal monopoly outputs are

$$q_i^m = \frac{d_i}{2b_i} \text{ and } q_j^m = \frac{d_j}{2b_j}$$

where the upper script "m" means "monopoly." The characteristic equation (6) is reduced to

$$\left(\lambda + \alpha_i^m e^{-\lambda\tau_i}\right)\left(\lambda + \alpha_j^m e^{-\lambda\tau_j}\right) = 0 \tag{8}$$

where

$$\alpha_k^m = K_k q_k^m (2b_k) > 0 \text{ for } k = i, j.$$

The characteristic equation for monopoly k is

$$\lambda + \alpha_k^m e^{-\lambda\tau_k} = 0 \text{ for } k = i, j \tag{9}$$

which has been studied in the literature.[1] There are two scenarios for instability, one is that a single real characteristic root becomes positive, and the other is that a pair of complex roots cross the imaginary axis at the same time. Since $\lambda \geq 0$ does not solve this equation, there is no possibility for the first scenario. Looking for a possibility of the second scenario, we assume $\lambda = i\omega$ with $\omega > 0$.[2] The characteristic equation (9) is transformed to

$$i\omega + \alpha_k^m (\cos\omega\tau_k - i\sin\omega\tau_k) = 0$$

and is separated into the real and imaginary parts

$$\alpha_k^m \cos\omega\tau_k = 0 \text{ and } \alpha_k^m \sin\omega\tau_k = \omega.$$

Adding the squares of these two equations yields

$$\omega = \alpha_k^m > 0.$$

Solving $\alpha_k \cos\omega\tau_k = 0$ for τ_k gives rise to the threshold value of τ_k,[3]

$$\tau_k^m(n) = \frac{1}{\alpha_k^m}\left(\frac{\pi}{2} + 2n\pi\right) \text{ for } k = i, j \text{ and } n \in \mathbb{N} \tag{10}$$

for which the characteristic equation has a root with zero real part.

We now turn our attention to the directions of the stability switch. We first assume the characteristic roots as continuous functions in terms of τ_k, $\lambda = \lambda(\tau_k)$

[1] See, for example, Matsumoto and Szidarovszky (2012, 2013).

[2] We arrive at the same result even if $\omega < 0$ is assumed.

[3] Solving the other equation, $\alpha_k \sin\omega\tau_k = \omega$ gives the same threshold value in a different form.

where τ_k is selected as the bifurcation parameter, and then determine the sign of the derivative of $\mathrm{Re}\left[\lambda(\tau_k)\right]$ at the point where $\lambda(\tau_k)$ is purely imaginary. Differentiation $\lambda = \lambda(\tau_k)$ with respect to τ_k and arranging the terms present

$$\left[1 - \alpha_k^m \tau_k e^{-\lambda\tau_k}\right] \frac{d\lambda}{d\tau_k} - \alpha_k^m \lambda e^{-\lambda\tau_k} = 0.$$

For convenience, we study $\left(d\lambda/d\tau_B\right)^{-1}$ instead of $d\lambda/d\tau_B$

$$\left(\frac{d\lambda}{d\tau_k}\right)^{-1} = \frac{1}{\alpha_k^m \lambda e^{-\lambda\tau_k}} - \frac{\tau_k}{\lambda}.$$

The second term is purely imaginary at $\lambda = i\omega$. Taking the real part and using $\alpha_k^m e^{-\lambda\tau_k} = -\lambda$ from (9), we have

$$\begin{aligned} \mathrm{Re}\left[\left(\frac{d\lambda}{d\tau_k}\right)^{-1}\Bigg|_{\lambda=i\omega}\right] &= \mathrm{Re}\left[\frac{1}{\lambda(-\lambda)}\Bigg|_{\lambda=i\omega}\right] \\ &= \frac{1}{\omega^2} > 0. \end{aligned}$$

Therefore, this inequality direction implies that the only crossing of the imaginary axis is from left to right as τ_k increases.

The positivity signals a potential Hopf bifurcation. The normal form theory and the center manifold theorem can verify the potentiality, however, we will numerically confirm it. To this end, we take the specific parameter values in the following numerical analysis unless otherwise mentioned:

Assumption 3. $K_i = K_j = 1,\ d_i = d_j = 4,\ b_i = b_j = 3/2,\ \gamma_{ij} = \gamma_{ji} = 1.$

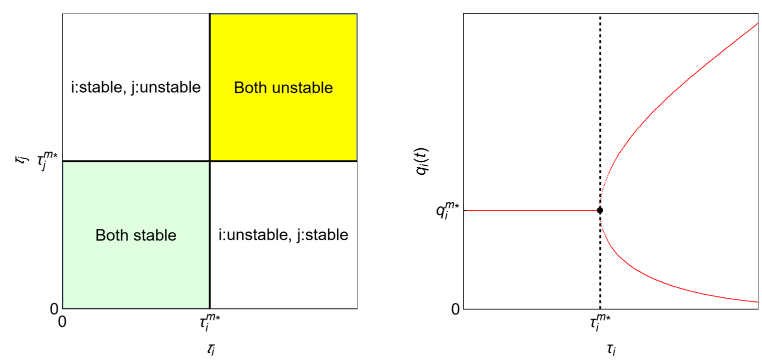
Taking $\gamma_{ij} = \gamma_{ji} = 0$ but retaining the other parameter values in Assumption 3, we depict the division of the (τ_1, τ_2) space for delay monopolies without spillovers. Both monopolies are stable in the light green region and unstable in the yellow region in Figure 1(A). One monopolistic firm is stable, and the other is unstable in the white region. Figure 1(B) presents a bifurcation diagram of regional monopoly i's output with respect to $\tau_i \in [0.3, 0.5]$. The black point is the bifurcation point of (τ_i^{m*}, q_i^{m*}).[4] The trajectory converges to the equilibrium q_i^{m*} for $\tau_i < \tau_i^{m*}$ and bifurcates to a limit cycle for $\tau_i > \tau_i^{m*}$ via a Hopf bifurcation. We can see that the Hopf bifurcation is super-critical, and the cycle gets larger as the value of τ_i increases further than τ_i^{m*}. Since the monopolies are symmetric, we have the same results for monopoly j. We summarize the results:

[4] Notice that each monopoly is independent. Regardless of the delay value of the other monopoly, the bifurcation point is the same where Assumption 3 implies

$$\tau_i^{m*} = \frac{\pi}{8} \simeq 0.3927 \text{ and } q_i^{m*} = \frac{4}{3}.$$

Theorem 2 *If Assumption 3 hold, and no spillovers, $\gamma_{ij} = \gamma_{ji} = 0$, are assumed, then the production delay is harmless for $\tau_k < \tau_k^{m*}$ and can destabilize the equilibrium to give rise to the birth of a limit cycle for $\tau \geq \tau_k^{m*}$ where τ_k^{m*} is the minimum value of $\tau_k^m(n)$ obtained for $n = 0$,*

$$\tau_k^{m*} = \frac{\pi}{2\alpha_k^m} \text{ for } k = i, j.$$



(A) Region division (B) Bifurcation diagram

Figure 1. Delay monopolies with no spillovers, $\gamma_{ij} = \gamma_{ji} = 0$

After considering the production delay effect, we are now in a position to examine the information delay effect. To this purpose, going back to linear system (4), we assume $\tau_i = \tau_j = 0$ and denote the sum of the information delays as $\tau = \tau_{ij} + \tau_{ji}$. The characteristic equation (6) now has the form,

$$(\lambda + \alpha_i)(\lambda + \alpha_j) - \beta_i \beta_j e^{-\lambda\tau} = 0. \tag{11}$$

For $\lambda = i\omega$ with $\omega > 0$, as before, the characteristic equation is separated into the real and imaginary parts,

$$-\omega^2 + \alpha_i \alpha_j - \beta_i \beta_j \cos \omega\tau = 0$$

and

$$(\alpha_i + \alpha_j)\omega + \beta_i \beta_j \sin \omega\tau = 0.$$

Adding the squares of these equations presents a fourth-order equation or a biquadratic equation in ω,

$$\omega^4 + \left(\alpha_i^2 + \alpha_j^2\right)\omega^2 + (\alpha_i \alpha_j)^2 - \left(\beta_i \beta_j\right)^2 = 0,$$

and its solutions of ω^2 are

$$\omega_\pm^2 = \frac{-\left(\alpha_i^2 + \alpha_j^2\right) \pm \sqrt{\left(\alpha_i^2 + \alpha_j^2\right)^2 - 4\left[(\alpha_i \alpha_j)^2 - \left(\beta_i \beta_j\right)^2\right]}}{2} < 0$$

where the inequality is due to the positive discriminant being less than $\left(\alpha_i^2+\alpha_j^2\right)$ due to one of the stability conditions, $\alpha_i\alpha_j-\beta_i\beta_j>0$. Hence, there is no $\omega>0$ satisfying $\lambda=i\omega$. Therefore, no stability switches occur for any $\tau\geq 0$; the information delays are *harmless*. We summarize the second results:

Theorem 3 *If Assumption 1 holds, and no production delays are assumed,* $\tau_i=\tau_j=0$*, then the information delays are harmless for any* $\tau=\tau_{ij}+\tau_{ji}$.

In the following, we consider the effects caused by different types of delays on dynamics. The characteristic equation (6) involves four different delays, τ_i, τ_j, $\tau_i+\tau_j$ and $\tau_{ij}+\tau_{ji}$. It is too complicated to deal with the four delays simultaneously. To proceed, we make various simplifying assumptions and then closely look at how the delays affect the otherwise stable equilibrium:

Dynamics I: One-way independency in which assuming $\beta_i=0$ or $\beta_j=0$ reduces (6) to two one-delay equations with τ_i or τ_j.

Dynamics II: Symmetric monopolies in which assuming $\tau_i=\tau_j=\tau_A$ and $\tau_{ij}=\tau_{ji}=\tau_B$ reduces (6) to a two-delay equation with τ_A and τ_B.

Dynamics III: Four delays with identical sums of two delays in which assuming $\tau_i+\tau_j=\tau_{ij}+\tau_{ji}$ reduces (6) to an equation with three delays, τ_i, τ_j, $\tau_i+\tau_j$.

3 Dynamics I: One-way Dependency

As seen above, a regional monopoly behaves as a pure monopoly if there are no spillovers. This section introduces one-way spillover from monopoly j to monopoly i to see how the spillover affects dynamics. With $\gamma_{ij}>0$ and $\gamma_{ji}=0$, the dynamical system (3) becomes asymmetric,

$$\begin{aligned}\dot{q}_i(t)&=K_iq_i(t)\left[d_i-2b_iq_i(t-\tau_i)-\gamma_{ij}q_j(t)\right],\\ \dot{q}_j(t)&=K_jq_j(t)\left[d_j-2b_jq_j(t-\tau_j)\right].\end{aligned}\tag{12}$$

The positive stationary point of system (12) is a pair of $\bar{q}_i^*$ and $\bar{q}_j^*$ such as

$$\bar{q}_i^*=q_i^{m*}-\gamma_{ij}\frac{d_j}{4b_ib_j}<q_i^{m*}\text{ and }\bar{q}_j^*=\frac{d_j}{2b_j}=q_j^{m*}.\tag{13}$$

The spillover has the effect of decreasing the equilibrium output of monopoly i. This is a natural consequence because a positive γ_{ij} causes an increase in the marginal production cost. The characteristic equation (6) with $\beta_j=0$ (i.e., $\gamma_{ji}=0$) is reduced to

$$\left(\lambda+\bar{\alpha}_ie^{-\lambda\tau_i}\right)\left(\lambda+\bar{\alpha}_je^{-\lambda\tau_j}\right)=0\tag{14}$$

with

$$\bar{\alpha}_i=K_i\bar{q}_i^*(2b_i)<\alpha_i^m\text{ and }\bar{\alpha}_j=K_j\bar{q}_j^*(2b_i)=\alpha_j^m.$$

The characteristic equation of monopoly k is

$$\lambda + \bar{\alpha}_k e^{-\lambda \tau_k} = 0 \text{ for } k = i, j,$$

which is formally the same as (9). Following Theorem 2, we can determine the threshold values,

$$\bar{\tau}_i^* = \frac{\pi}{2\bar{\alpha}_i} > \tau_i^{m*} \text{ and } \bar{\tau}_j^* = \frac{\pi}{2\bar{\alpha}_j} = \tau_j^{m*}. \tag{15}$$

Notice that the spillover $\gamma_{ij} > 0$ first gives rise to $\bar{q}_i^* < q_i^{m*}$ and then leads to $\bar{\alpha}_i < \alpha_i^m$, which results in the larger threshold value of the production delay, $\bar{\tau}_i^* > \tau_i^{m*}$. Hence, the spillover has a stabilizing effect in the sense that it extends the stability region.

With $\bar{\tau}_i^*$ and $\bar{\tau}_j^*$, we can validate whether the equilibria are stabile or unstable. They are locally asymptotically stable if

$$\tau_i < \bar{\tau}_i^* \text{ and } \tau_j < \bar{\tau}_j^*$$

and unstable if

$$\tau_i > \bar{\tau}_i^* \text{ and } \tau_j > \bar{\tau}_j^*.$$

It is also clear that the equilibrium output of monopoly i is unstable, and that of monopoly j is locally asymptotically stable if

$$\tau_i > \bar{\tau}_i^* \text{ and } \tau_j < \bar{\tau}_j^*.$$

We now turn attention to the destabilizing effect of the spillover. For the remaining delay combination,

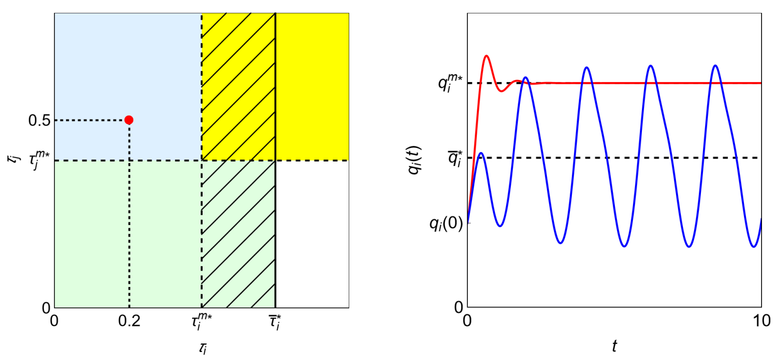
$$\tau_i < \bar{\tau}_i^* \text{ and } \tau_j > \bar{\tau}_j^*,$$

monopoly j's equilibrium is unstable when monopoly j has a larger delay value than its threshold value, $\bar{\tau}_j^*$. This instability of monopoly j spills over to monopoly i thought γ_{ij} and destabilizes the equilibrium of monopoly i, although, monopoly i has a smaller delay value than its threshold value, $\tau_i < \bar{\tau}_i^*$. This last result is due to the destabilizing effect of the spillover.

Assuming $\gamma_{ij} > 0$ and $\gamma_{ji} = 0$, we numerically confirm that the spillover has a doubl-edged effect: the stabilizing effect for enlarging the stability region and the destabilizing effect for transfomring stability to instability. The stabilizing effect of the spillover causes the rightward shift of the vertical dotted line at τ_i^{m*} to the vertical solid line at $\bar{\tau}_i^*$ (i.e., $\bar{\tau}_i^* > \tau_i^{m*}$), implying an expansion of the shaded light green region in Figure 2(A). The destabilizing effect is visualized in Figure 2(B). There, choosing the delay values at the red point of $\tau_i = 0.2 < \bar{\tau}_i^*$ and $\tau_j = 0.5 > \bar{\tau}_j^*$ in Figure 2(A) and taking constant initial functions $\phi_k(t) = 0.5$ for $t \leq 0$, we run the delay system (12) for $0 \leq t \leq 10$ twice, the first run with $\gamma_{ij} = 0$, and the second with $\gamma_{ij} = 1$. The first result is depicted in red and the second in blue. The two trajectories start at the same point, $q_i(0)$: the red trajectory with no spillover converges to the stationary point, q_i^{m*} because

$\tau_i < \bar{\tau}_i^*$, whereas, after initial disturbances, the blue trajectory keeps oscillating around the stationary point, $\bar{q}_i^*$. The destabilization in j's region is imported to i's region via $\gamma_{ij} > 0$. Notice that the production delay of $\tau_i = 0.2$ is positive but harmless. Accordingly. the destabilizing effect dominates the stabilizing effect. This dominance takes place for any pairs of the delays in the light blue and the shaded yellow regions in Figure 2(A). The instability spillover from monopoly j can be a source of cyclic behavior of otherwise stable monopoly i. We summarize these spillover effects as follow:

Theorem 4 *The one-way spillover (i.e., $\gamma_{ij} > 0$ and $\gamma_{ji} = 0$) has the stabilizing effect and the destabilizing effect on monopoly i: (i) the stabilizing effect enlarges the stability (i.e., green) region by shifting the partition line rightward; (ii) the destabilizing effect enlarges the instability (i.e., the light blue and shaded yellow) regions by making otherwise stable equilibrium unstable and generating cyclic oscillations.*



(A) Space division (B) Cyclic oscillations

Figure 2. Spillover has stabilizing and destabilizing effects

4 Dynamics II: Symmetric Monopolies

In this section, symmetric monopolies are considered, when the model parameters are

$$a_i = a_j,\ b_i = b_j,\ c_i = c_j,\ \gamma_{ij} = \gamma_{ji},\ K_i = K_j.$$

Then clearly we have

$$d_i = d_j \text{ and } q_i^* = q_j^*.$$

Furthermore,

$$\alpha_i = \alpha_j = \alpha \text{ and } \beta_i = \beta_j = \beta.$$

We retain the positive production and information delays, $\tau_i > 0,\ \tau_j > 0,\ \tau_{ij} > 0$ and $\tau_{ji} > 0$ but confine attention to the special case,

$$\tau_i = \tau_j = \tau_A > 0 \text{ and } \tau_{ij} = \tau_{ji} = \tau_B > 0. \tag{16}$$

Under these simplifications, monopolies are symmetric, and the characteristic equation (6) becomes

$$\left(\lambda+\alpha e^{-\lambda\tau_A}\right)^2-\left(\beta e^{-\lambda\tau_B}\right)^2=0 \tag{17}$$

which is factorized as

$$\left(\lambda+\alpha e^{-\lambda\tau_A}+\beta e^{-\lambda\tau_B}\right)\left(\lambda+\alpha e^{-\lambda\tau_A}-\beta e^{-\lambda\tau_B}\right)=0.$$

For convenience, we consider the two equations having the two delays together,

$$\lambda+\alpha e^{-\lambda\tau_A}\pm\beta e^{-\lambda\tau_B}=0. \tag{18}$$

Now we follow the method discussed in Appendix A.3 of Matsumoto and Szidarovszky (2018), some of which are based on the idea of Gu et al. (2005). Dividing both sides of (18) by λ presents a new form,

$$1+a_1(\lambda)e^{-\lambda\tau_A}+a_2(\lambda)e^{-\lambda\tau_B}=0 \tag{19}$$

where

$$a_1(\lambda)=\frac{\alpha}{\lambda} \text{ and } a_2(\lambda)=\pm\frac{\beta}{\lambda}. \tag{20}$$

We look for pure complex solutions, $\lambda=i\omega$ with $\omega>0$ since $\lambda=0$ does not solve equation (17). Notice that the three complex vectors of (19),

$$1,\ a_1(i\omega)e^{-i\omega\tau_A} \text{ and } a_2(i\omega)e^{-i\omega\tau_B},$$

form a triangle in the complex plane as shown in Figure 3 if and only if the following triangle conditions hold,

$$|a_1(i\omega)|+|a_2(i\omega)|\geq 1,$$

and

$$-1\leq|a_1(i\omega)|-|a_2(i\omega)|\leq 1.$$

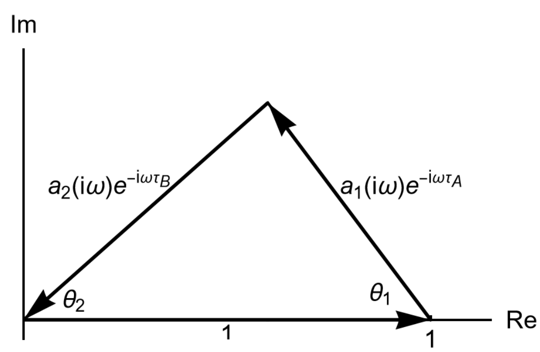


Figure 3. A triangle constructed by three vectors

Substituting $\lambda=i\omega$ into $a_1(\lambda)$ and $a_2(\lambda)$ in (19) presents

$$a_1(i\omega)=-i\frac{\alpha}{\omega} \text{ so } |a_1(i\omega)|=\frac{\alpha}{\omega},$$

and

$$a_2(i\omega) = \pm i\frac{\beta}{\omega} \text{ so } |a_2(i\omega)| = \frac{\beta}{\omega}.$$

The triangle conditions are written as

$$0 < \alpha - \beta \leq \omega \leq \alpha + \beta$$

where $\alpha - \beta > 0$ is due to Assumption 2. The law of cosine can be used to find angles θ_1 and θ_2:

$$\theta_1(\omega) = \cos^{-1}\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right),$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{1 + |a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2\,|a_2(i\omega)|}\right).$$

Since the triangle can be placed above and below the horizontal axis, we have

$$\arg[a_1(i\omega)] - \omega\tau_A \pm \theta_1(\omega) = \pi$$

and

$$\arg[a_2(i\omega)] - \omega\tau_B \mp \theta_2(\omega) = \pi,$$

implying that the threshold delay values are obtained as follows:

$$\tau^{\pm}_{A,m}(\omega) = \frac{1}{\omega}\left[\arg[a_1(i\omega)] + (2m-1)\pi \pm \theta_1(\omega)\right] \text{ for } m \in \mathbb{Z}_+$$

and

$$\tau^{\mp}_{B,n}(\omega) = \frac{1}{\omega}\left[\arg[a_2(i\omega)] + (2n-1)\pi \mp \theta_2(\omega)\right] \text{ for } n \in \mathbb{Z}_+$$

where, for $\omega > 0$,

$$\arg[a_1(i\omega)] = -\frac{\pi}{2} \text{ and } \arg[a_2(i\omega)] = \begin{cases} \dfrac{\pi}{2} \text{ if } a_2(i\omega) = i\dfrac{\beta}{\omega}, \\[2ex] -\dfrac{\pi}{2} \text{ if } a_2(i\omega) = -i\dfrac{\beta}{\omega}. \end{cases}$$

Let us denote the interval of ω satisfying the triangle condition by $\Omega = [\alpha - \beta, \alpha + \beta]$. We summarize the results obtained so far: the construction of the switching curves by the pairs of τ_A and τ_B is described in the following:

Theorem 5 *The switching curves are constructed by the pairs of τ_A and τ_B belonging to $T^{\pm}_{m,n}(\omega)$ where*

$$T^{\pm}_{m,n}(\omega) = \{\left(\tau^{\pm}_{A,m}(\omega), \tau^{\mp}_{B,n}(\omega)\right)\Big|\ \omega \in \Omega,\ m \in \mathbb{Z}_+,\ n \in \mathbb{Z}_+\}.$$

Assumption 3 implies

$$q_i^* = q_j^* = 1,\ \alpha_i = \alpha_j = 3 \text{ and } \beta_i = \beta_j = 1.$$

Since $\gamma_{ij} = \gamma_{ji} > 0$, we see the enlargement of the light green stability and the shaded yellow regions due to the stability effect of the spillover (i.e., $\tau_i^{m*} \to \tau_A^*$ and $\tau_j^{m*} \to \tau_B^*$). In addition, if there are no information delays, we will see that, due to the destabilizing effect of the spillover, the equilibrium points are unstable in the white regions in the upper-left and lower-right parts. However, we have positive information delays (i.e., $\tau_{ij} = \tau_{ji} = \tau_B > 0$). Although it is shown in Theorem 3 that the information delay is harmless when the production delays are zero,[5] the positive information delays make essential contributions constructing the stability switching curve, as shown in Theorem 5. In particular, in Figure 4(A), the blue segment is described by $\left\{\tau^{\pm}_{A,m}(\omega), \tau^{\mp}_{B,n}(\omega)\right\}$ for $m = 0, 1$ and $n = 1$ when $a_2(i\omega) = -i\beta/\omega$, and the red segment by $\left\{\tau^{\pm}_{A,m}(\omega), \tau^{\mp}_{B,n}(\omega)\right\}$ for $m = 1$ and $n = 0, 1$ when $a_2(i\omega) = i\beta/\omega$.[6] The stability switching curve is the left envelope of these red and blue curves. If a point (τ_A, τ_B) crosses the left envelope, then the corresponding pair of characteristic roots cross the imaginary axis to the right from the left. Thus, stability switching might occur. As a result, the stability in the white region to the left of the stability switching curve is regained. This is because the stabilizing effect of the information delay dominates the destabilizing effect of the spillover. On the other hand, some parts of the enlarged light green region in the lower-left lose stability due to the stronger production delay effect.

The dotted vertical line at $\tau_A^0 = 0.385$ crosses the blue segment at points a and b and the red segment at point c,[7]

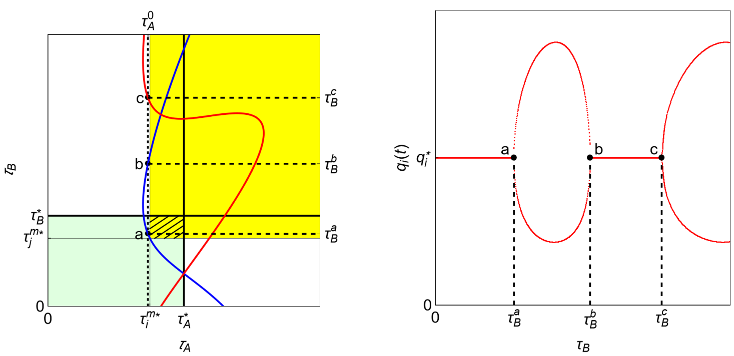
$$\tau_B^a \simeq 0.418,\ \tau_B^b \simeq 0.824 \text{ and } \tau_B^c \simeq 1.205.$$

Suppose τ_B keeps moving upward along this vertical dotted line. The stability is switched to instability (i.e., stability loss and the birth of a limit cycle) at points a and c, while the instability is switched to stability (i.e., stability gain and the demise of a limit cycle) at point b. Figure 4(B) illustrates the corresponding bifurcation diagram with respect to τ_B, showing the birth and death of a Hopf cycle at those points from a different viewpoint. Between these points, the destabilizing spillover effect is large enough to generate periodic behavior for $\tau_B \in (\tau_B^a, \tau_B^b)$, and the stabilizing information delay effect is large enough to make the stationary point stable for $\tau_B \in [0, \tau_B^a) \cup (\tau_B^b, \tau_B^c)$.

[5] The harmless information delay for $\tau_A = 0$ is graphically established in Figure 4(A) as the vertical axis at $\tau_A = 0$ is located to the left of the stability switching curve.

[6] Increasing the values of m and n constructs a series of sprial-like curves.

[7] Since $\tau_i^{m*} = \tau_j^{m*} \simeq 0.3927$, the partition line at τ_i^{m*} is located to the right of the dotted line at τ_A^0, and the horizontal partition line at τ_j^{m*} is below the dotted horizontal line at τ_B^a.



(A) Space division (B) Bifurcation diagram

Figure 4. Multiple stability switches

5 Dynamics III: Identical Sums of Two Delays

In this section, we consider the case in which the sum of the production delays is equal to the sum of the information delays, $\tau_i + \tau_j = \tau_{ij} + \tau_{ji}$. Under this condition, the characteristic equation (6) is transformed to

$$\left(\lambda + \alpha_i e^{-\lambda\tau_i}\right)\left(\lambda + \alpha_j e^{-\lambda\tau_j}\right) - \beta_i\beta_j e^{-\lambda(\tau_i+\tau_j)} = 0 \tag{21}$$

which is rewritten as

$$P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_i} + P_2(\lambda)e^{-\lambda\tau_j} + P_3(\lambda)e^{-\lambda(\tau_j+\tau_j)} = 0 \tag{22}$$

where there are three different delays, $\tau_i,\ \tau_j$ and $\tau_j + \tau_j$,

$$P_0(\lambda) = \lambda^2,\ P_1(\lambda) = \alpha_i\lambda,\ P_2(\lambda) = \alpha_j\lambda,\ P_3(\lambda) = \alpha_i\alpha_j - \beta_i\beta_j. \tag{23}$$

Since equation (22) has a more general form than equation (19), the method to solve (19) is not applicable to (22). Applying the results obtained in Appendix A.3 of Matsumoto and Szidarivszky (2018), some of which are based on the ideas of Lin and Wang (2012), we will solve the characteristic equation (22). Before proceeding, we check the following conditions to guarantee that (22) can be the characteristic equation of a delay system and to exclude some trivial cases:

(a) There are a finite number of characteristic roots on C_+ (i.e., $\mathrm{Re}\,\lambda > 0$) when

$$\deg\left[P_0(\lambda)\right] \geq \max\left\{\deg\left[P_1(\lambda)\right],\ \deg\left[P_2(\lambda)\right],\ \deg\left[P_3(\lambda)\right]\right\}.$$

(b) The zero frequency $\lambda = 0$ is not a characteristic root with any τ_1 and τ_2,

$$P_0(0) + P_1(0) + P_2(0) + P_3(0) \neq 0.$$

(c) Polynomials $P_0(\lambda),\ P_1(\lambda),\ P_2(\lambda)$ and $P_3(\lambda)$ have no common roots.

(d) $P_k(\lambda)$ for $k = 0, 1, 2, 3$ satisfy

$$\lim \left[\left| \frac{P_1(\lambda)}{P_0(\lambda)} \right| + \left| \frac{P_2(\lambda)}{P_0(\lambda)} \right| + \left| \frac{P_3(\lambda)}{P_0(\lambda)} \right| \right] < 1.$$

If (a) is violated, then there are infinitely many roots with positive real roots, and no stability is obtained. If (b) is violated, then (21) holds for any $(\tau_i, \tau_j) \in \mathbb{R}_+^2$, and therefore the characteristic function is always unstable. Condition (c) ensures that the considered characteristic equation has the lowest degree and is irreducible. Condition (d) is to exclude large oscillations. Notice the following: (a) and (d) hold since $P_0(\lambda)$ is quadratic, while $P_1(\lambda)$ and $P_2(\lambda)$ are linear and $P_3(\lambda)$ is constant; (b) holds since $\alpha_i\alpha_j - \beta_i\beta_j > 0$ under Assumption 1; (c) is trivial. By condition (b), $\lambda = 0$ does not solve the characteristic equation (6) with $\tau_i + \tau_j = \tau_{ij} + \tau_{ji}$, we seek purely imaginary characteristic roots, to study stability switching.

Since roots of a real function always come in conjugate pairs, we assume $\lambda = i\omega$ with $\omega > 0$. Substituting this into (21), we get

$$\left(P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_i}\right) + \left(P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_i}\right)e^{-i\omega\tau_j} = 0 \tag{24}$$

Since $\left|e^{-i\omega\tau_j}\right| = 1$, we have

$$\left|P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_i}\right| = \left|P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_i}\right|$$

which is equivalent to

$$\left(P_0 + P_1 e^{-i\omega\tau_i}\right)\left(\bar{P}_0 + \bar{P}_1 e^{i\omega\tau_i}\right) = \left(P_2 + P_3 e^{-i\omega\tau_i}\right)\left(\bar{P}_2 + \bar{P}_3 e^{i\omega\tau_i}\right)$$

where argument $i\omega$ of P_k is omitted for notational simplicity, and the upper-bar means conjugate complex number. After arranging the terms, we have

$$|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2 = 2A_i(\omega)\cos\omega\tau_i - 2B_i(\omega)\sin\omega\tau_i \tag{25}$$

Here, $A_i(\omega)$ and $B_i(\omega)$ are defined as follows,

$$A_i(\omega) = \mathrm{Re}\left[P_2\bar{P}_3 - P_0\bar{P}_1\right] \text{ and } B_i(\omega) = \mathrm{Im}\left[P_2\bar{P}_3 - P_0\bar{P}_1\right]$$

where

$$P_2\bar{P}_3 - P_0\bar{P}_1 = i\alpha_i\omega\left[\frac{\alpha_j}{\alpha_i}\left(\alpha_i\alpha_j - \beta_i\beta_j\right) - \omega^2\right].$$

Hence,

$$A_i(\omega) = 0 \text{ and } B_i(\omega) = \alpha_i\omega\left[\frac{\alpha_j}{\alpha_i}\left(\alpha_i\alpha_j - \beta_i\beta_j\right) - \omega^2\right].$$

Assumption 2 implies $\alpha_i\alpha_j - \beta_i\beta_j > 0$, then

$$B_i(\omega) \gtreqless 0 \text{ according to } \omega \lesseqgtr \omega_0 = \sqrt{\frac{\alpha_j}{\alpha_i}\left(\alpha_i\alpha_j - \beta_i\beta_j\right)}. \tag{26}$$

Assumption 3 leads to $\omega_0 = 2\sqrt{2}$. Henceforth, we consider two cases, $B_i(\omega) \neq 0$ and $B_i(\omega) = 0$, separately.

5.1 $B_i(\omega) \neq 0$

We assume $B_i(\omega) \neq 0$ (i.e., $\omega \neq \omega_0$) in this subsection and denote the argument of $iB_i(\omega)$ by $\phi_i(\omega)$. Formally, we can rewrite $B_i(\omega)$ as

$$B_i(\omega) = \sqrt{B_i(\omega)^2} \sin(\phi_i(\omega))$$

where

$$\phi_i(\omega) = \arg[iB_i(\omega)] = \begin{cases} \pi/2 \text{ if } B_i(\omega) > 0 \text{ for } \omega < \omega_0, \\ \\ -\pi/2 \text{ if } B_i(\omega) < 0 \text{ for } \omega > \omega_0. \end{cases}$$

Hence, the right-hand side of (25) is written as

$$2\sqrt{B_i(\omega)^2}\left[\cos(\phi_i(\omega))\cos(\omega\tau_i) - \sin(\phi_i(\omega))\sin(\omega\tau_i)\right]$$

which the Addition theorem simplifies as

$$2\sqrt{B_i(\omega)^2}\cos(\phi_i(\omega) + \omega\tau_i).$$

Let $F_i(\omega)$ denote the left-hand side of (25),

$$F_i(\omega) = |P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2.$$

There exists a $\tau_i > 0$ with which (25) is written as

$$F_i(\omega) = 2\sqrt{B_i(\omega)^2}\cos(\phi_i(\omega) + \omega\tau_i) \tag{27}$$

if and only if the following condition holds

$$|F_i(\omega)| \leq 2\sqrt{B_i(\omega)^2}. \tag{28}$$

It is more convenient to manage if we square both sides of (28) to have the following form,

$$[F_i(\omega)]^2 - 4B_i(\omega)^2 \leq 0.$$

Since the monopolies are symmetric, $\alpha_i = \alpha_j = \alpha$ and $\beta_i = \beta_j = \beta$, the left-hand side of (25) is factorized as

$$(\omega - (\alpha - \beta))(\omega - (\alpha + \beta))(\omega + (\alpha - \beta))(\omega + (\alpha + \beta))\left(\omega^2 - (\alpha^2 - \beta^2)\right)^2$$

and $\alpha > \beta > 0$ by Assumption 2. The last three factors are positive. Therefore, if $\omega \in \Omega = (\alpha - \beta, \alpha + \beta)$, then we have

$$[F_i(\omega)]^2 - 4B_i(\omega)^2 < 0.$$

The condition holds with equality when $\omega = \alpha - \beta$ or $\omega = \alpha + \beta$. We can define some continuous function $\psi_i(\omega)$ such that

$$\cos(\psi_i(\omega)) = \frac{F_i(\omega)}{2\sqrt{B_i(\omega)^2}} \text{ for } \psi_i(\omega) \in [0, \pi]. \tag{29}$$

Comparing (27) with (29), we have the following relation,

$$\cos(\psi_i(\omega)) = \cos(\phi_i(\omega) + \omega\tau_i)$$

or the threshold value of τ_i is obtained as

$$\tau^{\pm}_{i,m}(\omega) = \frac{\pm\psi_i(\omega) - \phi_i(\omega) + 2m\pi}{\omega} \text{ for } m \in \mathbb{Z}_+. \tag{30}$$

In the same way, we can determine the threshold value of τ_j. First, we rewrite equation (21) as

$$P_0(\lambda) + P_2(\lambda)e^{-\lambda\tau_j} + \left(P_1(\lambda) + P_3(\lambda)e^{-\lambda\tau_j}\right)e^{-\lambda\tau_i} = 0. \tag{31}$$

Repeating the procedure for deriving the threshold value of τ_i with equation (30), we obtain the form of τ_2 for which the characteristic equation has purely imaginary roots,

$$\tau^{\pm}_{j,n}(\omega) = \frac{\pm\psi_j(\omega) - \phi_j(\omega) + 2n\pi}{\omega} \text{ for } n \in \mathbb{Z}_+ \tag{32}$$

where

$$\cos\left(\psi_j(\omega)\right) = \frac{|P_0|^2 + |P_2|^2 - |P_1|^2 - |P_3|^2}{2\sqrt{B_j(\omega)^2}} \text{ for } \psi_j(\omega) \in [0, \pi],$$

$$A_j(\omega) = 0 \text{ and } B_j(\omega) = \alpha_j\omega\left(\frac{\alpha_i}{\alpha_j}\left(\alpha_i\alpha_j - \beta_i\beta_j\right) - \omega^2\right)$$

$$\phi_j(\omega) = \arg[iB_j(\omega)] = \begin{cases} \pi/2 \text{ if } B_j(\omega) > 0 \text{ for } \omega < \omega_0, \\ -\pi/2 \text{ if } B_j(\omega) < 0 \text{ for } \omega > \omega_0. \end{cases}$$

It is also confirmed that

$$B_j(\omega) = \sqrt{B_j(\omega)^2}\sin\left(\phi_j(\omega)\right)$$

and

$$\left||P_0|^2 + |P_2|^2 - |P_1|^2 - |P_3|^2\right| \le 2\sqrt{B_j(\omega)^2} \text{ for } \omega \in \Omega.$$

Summarizing the results, we have the following:

Theorem 6 *The following pairs of delays construct the set of all switching curves*

$$\left\{\left(\tau^{\pm}_{i,m}(\omega),\ \tau^{\mp}_{j,n}(\omega)\right) \;\middle|\; \omega \in \Omega\right\} \textit{ for } m \in \mathbb{N} \textit{ and } n \in \mathbb{N},$$

where

$$\tau^{\pm}_{i,m}(\omega) = \frac{\pm\psi_i(\omega) - \phi_i(\omega) + 2m\pi}{\omega},$$

$$\tau^{\mp}_{j,n}(\omega) = \frac{\mp\psi_j(\omega) - \phi_j(\omega) + 2n\pi}{\omega}$$

and

$$\Omega = \{\omega \mid |F_k(\omega)| \le 2\sqrt{B_k(\omega)^2} \le 0 \textit{ for } k = i, j\}.$$

5.2 $B_i(\omega) = 0$

ω_0 is defined in (26) and the positive solution of $B_i(\omega) = 0$. Then

$$A_i(\omega_0) = B_i(\omega_0) = 0 \Longrightarrow P_2\bar{P}_3 = P_0\bar{P}_1$$

The right-hand side of (25) is 0 with any τ_i, and

$$|P_0|^2 + |P_1|^2 = |P_2|^2 + |P_3|^3 .$$

Therefore, all $\tau_i \in \mathbb{R}_+$ are solution of (25) for $\omega = \omega_0$. From (24), we obtain τ_j as function of τ_i,

$$e^{-i\omega\tau_j} = -\frac{P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_i}}{P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_i}}. \tag{33}$$

An explicit form of τ_j is derived as follows.[8] First, Euler's formula replaces the exponential terms with the trigonometric terms. Equation (33) is rewritten as

$$\cos\omega\tau_j - i\sin\omega\tau_j = -\frac{\boldsymbol{a} + i\boldsymbol{b}}{\boldsymbol{c} + i\boldsymbol{d}} \tag{34}$$

where

$$\boldsymbol{a} = \alpha_i\omega\sin\omega\tau_i - \omega^2 \text{ and } \boldsymbol{b} = \alpha_i\omega\cos\omega\tau_i$$

$$\boldsymbol{c} = \left(\alpha_i\alpha_j - \beta_i\beta_j\right)\cos\omega\tau_i \text{ and } \boldsymbol{d} = \alpha_j\omega - \left(\alpha_i\alpha_j - \beta_i\beta_j\right)\sin\omega\tau_i.$$

We rationalize the right-hand side of (34) by multiplying the denominator and numerator by the conjugate complex number of the denominator,

$$\cos\omega\tau_j - i\sin\omega\tau_j = -\frac{M}{D} - i\frac{N}{D}$$

or

$$\cos\omega\tau_j = -\frac{M}{D} \text{ and } \sin\omega\tau_j = \frac{N}{D}$$

where

$$M = \boldsymbol{ac} + \boldsymbol{bd},\ N = \boldsymbol{bc} - \boldsymbol{ad} \text{ and } D = \boldsymbol{c}^2 + \boldsymbol{d}^2.$$

Under Assumption 3,

$$M = \omega_0^2\cos\left(\omega_0\tau_i\right),$$

$$N = \omega_0\left(3(8+\omega_0^2) - 17\omega_0\sin\left(\omega_0\tau_i\right)\right),$$

$$D = 64 + 9\omega_0^2 - 48\omega_0\sin\left(\omega_0\tau_i\right)$$

and, for $\tau_i = 0$ and $\tau_i = 2\pi/\omega_0$,

$$-\frac{M}{D} = -\frac{1}{17} \simeq -0.05882 \text{ and } \frac{N}{D} = \frac{12\sqrt{2}}{17} \simeq 0.99827$$

which are denoted as y_0 and y_1, respectively.

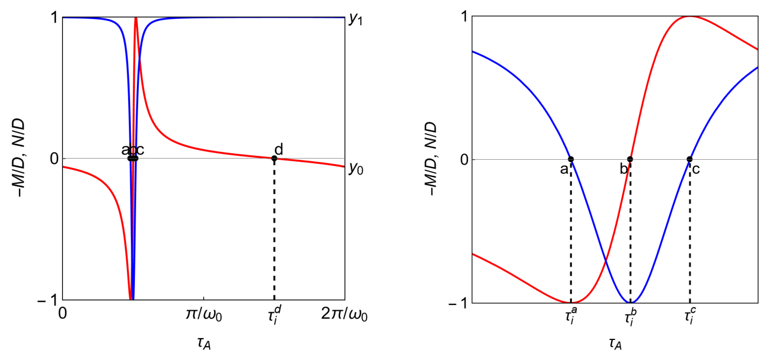
[8] In the same way, it is possible to derive $\tau_i(\tau_j)$ from (31).

Figure 5(A) illustrates the graphs of $-M/D$ and N/D in red and blue for $\tau_i \in [0, 2\pi/\omega_0]$, and Figure 5(B) is an enlargement of the congested part of Figure 5(A). The red curve intersects the horizontal axis twice at points b and d, at which $M = 0$ or $\cos(\omega_0 \tau_i) = 0$, implying that

$$\tau_i^b = \frac{\pi}{2\omega_0} \text{ and } \tau_i^d = \frac{3\pi}{2\omega_0}.$$

The blue curve also intersects the horizontal axis twice at points a and c at which $N = 0$, implying

$$\tau_i^a = \frac{1}{\omega_0} \sin^{-1}\left(\frac{12\sqrt{2}}{17}\right) \text{ and } \tau_i^c = \frac{1}{\omega_0}\left[\pi - \sin^{-1}\left(\frac{12\sqrt{2}}{17}\right)\right].$$



(A) $\tau_A \in [0, 2\pi/\omega_0]$ (B) Enlargement

Figure 5. Graphs of $-M/D$ in red and N/D in blue

Those four points divide the interval $[0, 2\pi/\omega_0]$ into five subintervals in which the signs of $\sin \omega_0 \tau_j$ and $\sin \omega_0 \tau_j$ are determined. Accordingly, the form of $\tau_j(\tau_i)$ in each subinterval is determined:

$$\cos \omega_0 \tau_j < 0,\ \sin \omega_0 \tau_j > 0 \text{ for } \tau_i \in [0, \tau_i^a) \Longrightarrow \tau_j(\tau_i) = \frac{1}{\omega_o} \cos^{-1}\left(-\frac{M}{D}\right),$$

$$\cos \omega_0 \tau_j < 0,\ \sin \omega_0 \tau_j < 0 \text{ for } \tau_i \in [\tau_i^a, \tau_i^b) \Longrightarrow \tau_j(\tau_i) = \frac{1}{\omega_o}\left[2\pi - \cos^{-1}\left(-\frac{M}{D}\right)\right],$$

$$\cos \omega_0 \tau_j > 0,\ \sin \omega_0 \tau_j < 0 \text{ for } \tau_i \in [\tau_i^b, \tau_i^c) \Longrightarrow \tau_j(\tau_i) = \frac{1}{\omega_o}\left[2\pi - \cos^{-1}\left(-\frac{M}{D}\right)\right],$$

$$\cos \omega_0 \tau_j > 0,\ \sin \omega_0 \tau_j > 0 \text{ for } \tau_i \in [\tau_i^c, \tau_i^d) \Longrightarrow \tau_j(\tau_i) = \frac{1}{\omega_o} \cos^{-1}\left(-\frac{M}{D}\right),$$

$$\cos \omega_0 \tau_j < 0,\ \sin \omega_0 \tau_j > 0 \text{ for } \tau_i \in [\tau_i^d, 2\pi/\omega_0) \Longrightarrow \tau_j(\tau_i) = \frac{1}{\omega_o} \cos^{-1}\left(-\frac{M}{D}\right).$$

We summarize the results:

Theorem 7 *If $B_i(\omega_0) = 0$, then the locus of $(\tau_i, \tau_j(\tau_i))$ for $\tau_i \in [0, 2\pi/\omega_0]$ is the stability switching curve where*

$$\tau_j(\tau_i) = \frac{1}{\omega_0}\cos^{-1}\left(-\frac{M}{D}\right) \text{ for } \tau_i \in [0, \tau_i^a) \cup [\tau_i^c, 2\pi/\omega_0]$$

and

$$\tau_j(\tau_i) = \frac{1}{\omega_0}\left[2\pi - \cos^{-1}\left(-\frac{M}{D}\right)\right] \text{ for } \tau_i \in [\tau_i^a, \tau_i^c)$$

In Figure 6(A), as in Figure 4(A), the stabilizing spillover effects make the rightward and upward shifts of the partition lines, leading to the expansion of the stability region (i.e., two light green rectangles and the shaded yellow squares), and the equilibrium points are unstable in the white regions due to the destabilizing spillover effects if there are no information delays. The egg-shaped closed curve consists of the blue and red segments. The loci of $\left(\tau_{1,m}^{+}(\omega), \tau_{2,n}^{-}(\omega)\right)$ with $m = 0$ and $n = 1$ and $\left(\tau_{1,m}^{-}(\omega), \tau_{2,n}^{+}(\omega)\right)$ with $m = n = 0$ for $\omega \in \Omega$ constructs the blue segment, while the loci of $\left(\tau_{1,m}^{\pm}(\omega), \tau_{2,n}^{\mp}(\omega)\right)$ for $m = 0$ and $n = 0$ constructs the red segment. The upper convex-shaped and lower concave-shaped black loci of $(\tau_i, \tau_j(\tau_i))$ for $\tau_i \in [0, 2\pi/\omega_0]$ are other segments of the stability switching curve. The stability region includes the origin and is surrounded by black and red segments. In this example, the information delays also have stabilizing and destabilizing effects. The black curves divide the white regions into two subregions, and instability in the subregion attached to the enlarged rectangles is switched to stability. On the other hand, the red curve gouges the upper right portion of the stable region and small parts of the white region and destabilizes them. In this example. the increased regions are larger than the decreased regions. Hence, the information delays have the stabilizing effect all in all.

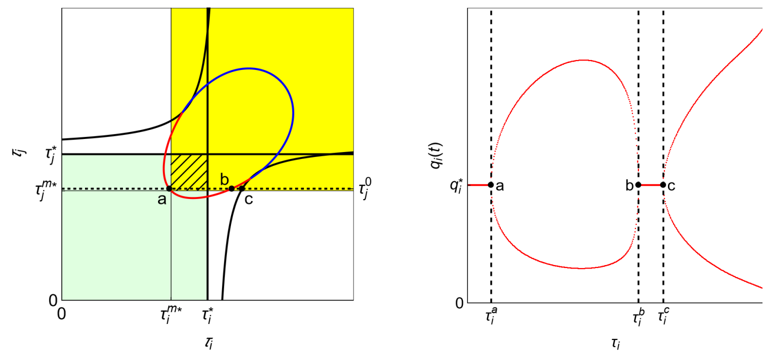
The horizontal dotted curve at $\tau_j^0 = 0.4$ crosses the red curve twice at points a and b, and the concave-shaped black curve at point c,[9]

$$\tau_i^a \simeq 0.386,\ \tau_i^b \simeq 0.61 \text{ and } \tau_i^c \simeq 0.648.$$

Looking at Figures 6(A) and 6(B), we see that, as the value of τ_i increases along the horizontal dotted line, the stationary point loses stability at τ_i^a, bifurcates to a limit cycle for $\tau_i \in (\tau_i^a, \tau_i^b)$, regains stability at τ_i^b, remains stable for

[9] Notice that the horizontal partition line at $\tau_j^{m*} \simeq 0.3927$ is located below the horizontal dotted line at τ_j^0.

$\tau_i \in (\tau_i^b, \tau_i^c)$ and then loses stability again at τ_i^c and get no more stability gain.



(A) Space division (B) Bifurcation diagram

Figure 6. Distorted stability region and multiple stability switching.

6 Concluding Remarks

We introduced and analyzed a dynamic model of regional monopolies having mutual spillovers. The model is simple but plausible for multi-regional monopolies in an industry: it contains enough model structure to describe the effects caused by interdependencies through spillovers, and it is quite natural in light of what we know about delays in producing output and acquiring information as a model for the evolution of mutual dependent regional monopolies. Since the delay dynamic model is a hybrid of discrete-time and continuous-time, it produces a richer assemblage of dynamics than the continuous-time model. For example, our analysis shows that due to the complex-shaped profile of the stability switching curveh, repeating stability loss and gain is possible for specific delay ranges. The production delay has a destabilizing effect, and the information delay can generate a stabilizing effect. Whether the model is stable depends on the relative magnitude of these opposite-signed effects, and which effect is dominant depends on the magnitudes between these delays. However, the larger delay does not necessarily provide a larger magnitude.

Although we have undertaken a rather thorough investigation of the dynamics under various combinations of four delays, τ_i, τ_j, $\tau_i + \tau_j$ and $\tau_{ij} + \tau_{ji}$, our study has not been exhaustive. It may be too ambitious to attempt the case of four distinct delays due to a lack of appropriate mathematical methods. The number of regional monopolies is limited to two only for analytical simplicity. Only the cost-increasing effects are focused on in this study. Another kind of spillover can contribute to cost reduction. We hope to modify some of these limitations in a forthcoming paper.

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