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Abstract

Using a strategic bequest model, in which the parents present a bequest rule that relates the place of residence to the bequest distribution ratio to one of two children, this paper examines how the distribution of bequests among children and the location of each child are determined. The birth order of children has a different effect depending on the total amount of parental bequests. When bequests are small, the second child lives closer to the parents, and receives a larger part of the bequests than the first child. On the other hand, when bequests are large, the first child lives with the parents and receives a larger part of the bequests than the second child. The latter result is consistent with traditional Japanese family residential patterns, suggesting that they can be explained not only by cultural and social norms, but also by economic rationality.

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1. Introduction

When young adults seek a job after completing their education, they should choose where to live and work henceforth. While various factors affect their location choices, an important one should be the possibility of providing attention or care for their elderly parents in the future. This is because the cost of caregiving, which includes the time as well as the transportation cost, crucially depends on the distance between their own and their parents' residence. Such a factor will become more important in the location choices of young adults, as longevity proceeds and the number of potential caregivers decreases due to low fertility.

Considering a family that consists of parents and two children, Konrad *et al*. (2002) show that there can emerge an equilibrium where the first child locates far away from the parents and the second child locates next to the parents. In this location pattern, the first child does not provide attention and care for the parents and the second child provides all, because the distance between a child's location and that of his or her parents is crucial for how much attention or care he or she can provide for the parents. This implies that the first child, who makes a residential choice prior to his or her sibling, moves sufficiently far away to induce the second child to live close to the parents and take care of them. While Konrad *et al*. also show that this birth order effect is supported by an empirical analysis using German data, it is doubtful that this applies to the location pattern of siblings in Japan. Since the idea that the eldest son should inherit the family estate (land and housing) and take care of his parents has not disappeared due to the remnants of the prewar Japanese family system, it is fairly common for the eldest son or the first child to live with his or her parents in Japan[1]. While this behavior has been often explained by social and cultural norms in Japan, it can also involve economic rationality. This paper attempts to provide a rationale for this behavior.

Konrad *et al*. do not consider parental bequests given to children. In Japan, however, the proportion of land and housing in the amount of inherited property is high, and children living with their parents often inherit land and housing. From these facts, it can

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[1] Hamaaki *et al*.(2019) state that the family system is one of the factors contributing to the different patterns of bequest division in the Western countries and Japan (equal distribution among children in the Western countries, while unequal distribution is common in Japan). In the Japanese traditional family system, which originated in the Edo period and was legislated in the Meiji period (1868-1912), the eldest son was designated as the heir to the family line. The eldest son was obliged to live with his parents and support them in their old age in exchange for inheriting the family estate (land and housing). Even after the major revision of civil law in 1947, the cultural and social norms of this traditional family system remain in Japan, with more elderly people preferring to live with their eldest son than in the Western countries. Hamaaki *et al.* (2019) empirically show that a larger share of bequests is distributed to the heir of the family line in Japan.

be inferred that there is a close relationship between residential choices and inheritance. Using Japanese survey-based microdata, Hamaaki *et al*.(2019) show that more bequests are distributed to children who live with their parents (or surviving parent), which is in line with the strategic bequest motive hypothesis (Bernheim, Shleifer and Summers, 1985). In this paper, therefore, we introduce parental bequests motivated by the strategic bequest motive into the model. That is, in our model, the parents precommit to a bequest rule which conditions the division of bequests on the children's place of residence in order to induce the children to live with or near the parents and receive more filial attention.

The model consists of parents and two children, each of whom derives utility from their own disposable income and the aggregate of the attention that the two children give to the parents. The children's income depends on their location, increasing as they move away from their parents and closer to the center of the economy, and reaching a maximum when they reside in the center of the economy. The level of attention each child gives to the parents depends on the distance between the parents and the child, and decreases as that distance increases. The parents first decide whether or not to present a bequest rule which relates the place of residence to the bequest distribution ratio to the first child, and if not to the first child, next decide whether or not to present the bequest rule to the second child. When parents present a bequest rule to the first child, the first child chooses the location specified in it, and then the second child chooses the location that maximizes his or her utility. When parents present a bequest rule to the second child, the first child, who was not presented with the bequest rule, chooses the location that maximizes his or her utility, and then the second child chooses the location specified in the bequest rule.

The main result obtained in this paper is that bequest distribution among children and family location pattern differ depending on the total amount of parental bequests. Birth order plays an important role also in this paper, but the effects of birth order on the children's location and attention they give to the parents differ from Konrad *et al*. If the parents present a bequest rule to the first child, and the first child chooses to live with or near the parent accordingly, then the parents do not present a bequest rule to the second child. This case suggests that there can arise a location pattern with the first child locating closer to the parents than the second child. We will show that whether the parents present a bequest rule to the first or second child depends on the total amount of parental bequests in the following sections.

The first previous study dealing with family residential choice is Konrad *et al.* (2002) cited above. They consider a model in which parents and an only child (or two children) choose a place of residence, and the child then chooses a level of attention to the parents.

In the case of two children, they show that there arises an equilibrium in which the parents do not move, the first child lives far enough away from the parents, and the second child lives with the parents. In this equilibrium, the first child provides no attention and all attention is provided by the second child. Rainer and Siedler (2009) extend the model of Konrad *et al.* by assuming a situation in which children choose where to seek employment in addition to where to live and potential earnings are different depending on where to seek employment (earnings opportunities in the distant labor market are better than those in the parents' location). In their model, the parents are not the player of the game, and bargaining determines each child's share of attention to the parents in the case of two children. They show that, since bargaining power increases with higher income and with living farther away from the parents' location, both children may live farther away from the parents. While the above two studies do not take into account income transfers between parents and children such as bequests[2], the amounts of bequests a child receives from his or her parents are likely to affect the child's residential choice. Focusing on this point, we aim to examine how the distribution of bequests among children and the place of residence of each child are determined.

The rest of the paper is organized as follows. Section 2 presents the model. In Section 3, we examine the bequest rule that the parents present to their second child. In Section 4, we examine the bequest rule that the parents present to their first child. In Section 5, based on Sections 3 and 4, we examine which of the first and second children the parents present with the bequest rule, and show the relationship between the total amount of the parental bequests and the location pattern of the family. Section 6 concludes the paper.

2. Model

We consider a linear economy where the economic activity is made on the real line, and a representative family that consists of parents and two children (a first child and a second child). The parents live and raise their children at some place that is normalized to 0. We assume that the parents never move away from point 0 throughout all their lives.

After completing their schooling, the first and second children each choose a location $f,\ s$. The children are employed in the labor market in the region where they live and earn income $Y_k(k)\ (k = f,\ s)$ there. Their incomes depend on their locations and we make the following assumption: the maximum income is obtained at k^c and the income

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[2] Analyzing the location choice of siblings in Japan, Kureishi and Wakabayashi (2010) focus on childcare assistance adult children receive from their parents, rather than caregiving to their parents, as a determinant of their location choice. In Kureishi and Wakabayashi, bequests or any other intergenerational income transfers are not taken into account.

falls as they live farther from k^c, which represents the central business district in the linear economy. This implies that, when the children live in the same home or locality as their parents and become employed in the local labor market, their income would be less than if employed at k^c. We assume that the income function $Y_k(k)$ is linear with respect to location, and

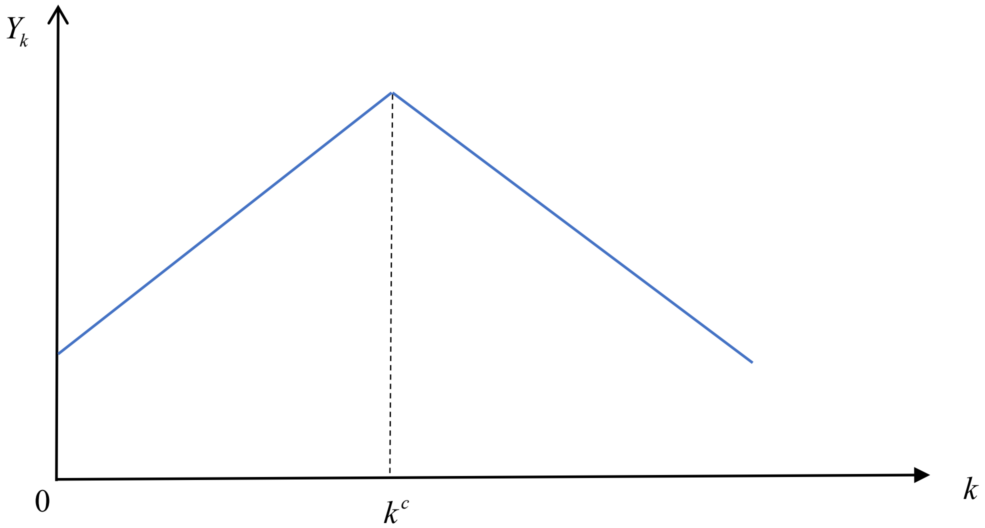
$$Y_k(0) > 0,\ Y_k'(k) > 0\ \textit{for all}\ k \in [0, k^c),\ Y_k'(k) < 0\ \textit{for all}\ k \in (k^c, \infty),$$

$$Y_{k-}'(k^c) = \lim_{k \to k_c - 0} [Y_k(k) - Y_k(k^c)]/(k - k^c) > 0,\ Y_{k+}'(k^c) = \lim_{k \to k_c + 0} [Y_k(k) - Y_k(k^c)]/(k - k^c) < 0,$$

$$Y_k''(k) = 0\ (k = f,\ s),$$

as illustrated in Figure 1.

Figure 1 Income function $Y_k(k)$



The parents need attention in old age, and the level of attention the children give to the parents depends on the distance between the parents and children because longer travel time means a greater cost of the visit. That is, since the parents' location is at point 0, $k\ (k = f,\ s)$ is equal to the distance between the parents' location and each child's location and the level of attention is given by $a(k)\ (k = f,\ s)$ with $da(k)/dk < 0\ (k = f,\ s)$. In addition, with an increase in k, $a(k)$ decreases sharply when the children's location is close to the parents', but does not decrease so much when the children live farther away from the parents. That is, we assume that $d^2 a(k)/dk^2 > 0\ (k = f,\ s)$.

The parents obtain utility from consumption and attention from their children.

$$U^p = u_p(Y_p - b) + v_p(a(f) + a(s)),$$

where Y_p is the parents' income and is constant, and b is the bequest. $u_p' > 0$, $u_p'' < 0$, $v_p'(a(f) + a(s)) > 0$ and $v_p''(a(f) + a(s)) < 0$ are assumed.

The utility function of each child is given by

$$U^k = u_k(Y_k(k) + \beta_k b) + v_k(a(f) + a(s)) - c(a_k) \ \ (k = f,\ s),$$

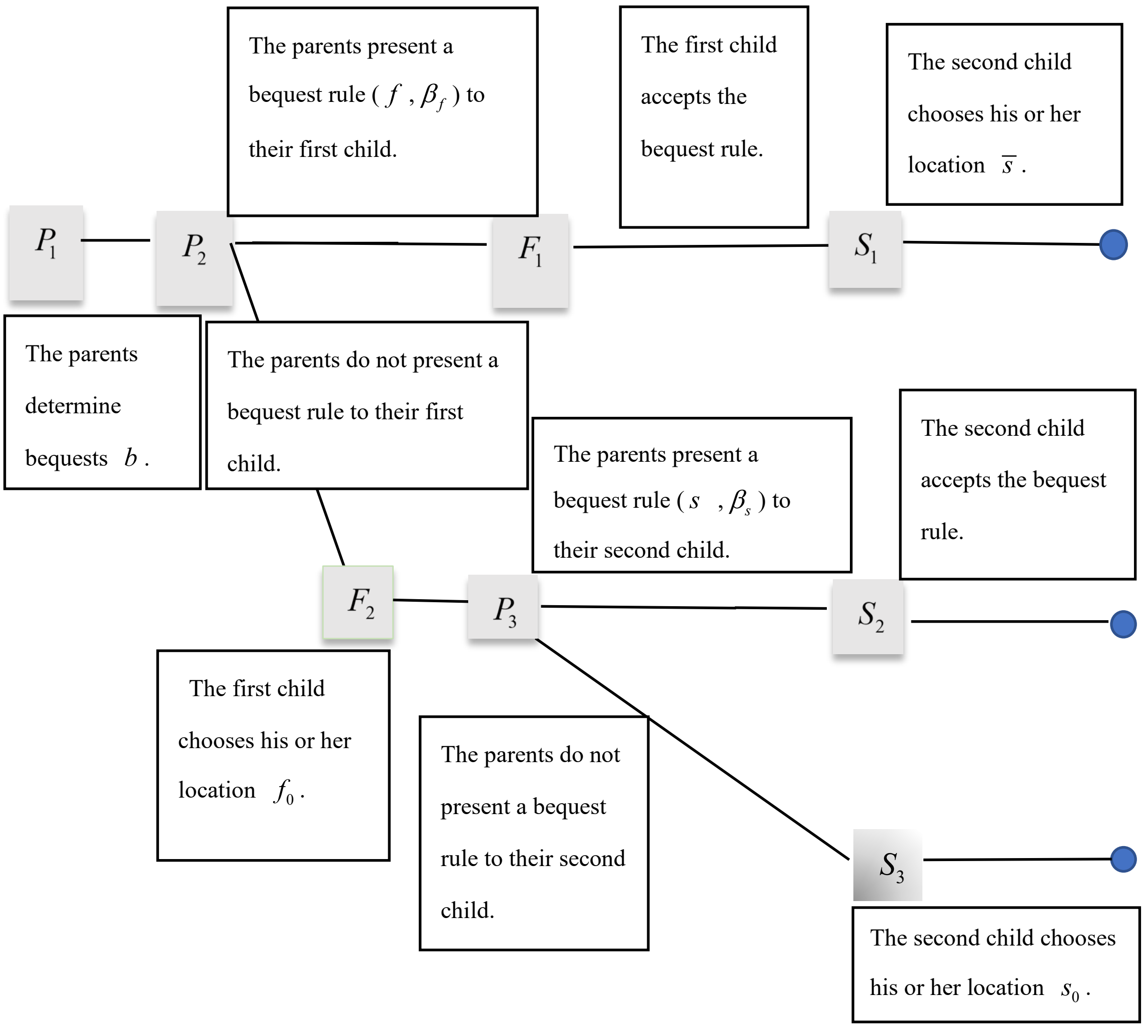
where $\beta_k (0 \le \beta_k \le 1)$ $(k = f,\ s)$ is the distribution ratio of the bequest b to the first and second child, respectively. $\beta_f + \beta_s = 1$ is assumed to be satisfied.

The children obtain utility from attention that the parents receive, but incur a cost for providing it: $c(a_k)$ $(k = f,\ s)$. We also assume that $u_k' > 0$, $u_k'' < 0$, $v_k'(a(f) + a(s)) > 0$, $v_k''(a(f) + a(s)) < 0$, $c'(a_k) > 0$ and $c''(a_k) = 0$ $(k = f,\ s)$.

The parents distribute their bequest to their two children based on strategic motives (Bernheim *et al*., 1985). That is, the parents try to use their bequest to have their children reside closer to them in order to obtain more attention.

The game tree of our model is shown in Figure 2. After determining the bequest b, the parents first decide whether or not to present the first child a bequest rule that relates the location f to the bequest distribution ratio β_f subject to the first child's participation constraint. If the parents present a bequest rule to the first child, the first child accepts it and the second child chooses the location that maximizes his or her utility. On the other hand, if the parents do not present a bequest rule to the first child, the first child resides in the location that maximizes his or her own utility. The parents then decide whether or not to present the second child a bequest rule that relates the location s to the bequest distribution ratio β_s subject to the participation constraint of the second child. If the parents present the second child a bequest rule, the second child accepts it. If the parents do not present a bequest rule to the second child, the second child chooses the location that maximizes his or her utility.

Figure 2 The extensive form of the game between the parents and children



3. Bequest rule the parents present to the second child

3-1 Second child's participation constraint

In this section, we examine the bequest rule the parents present to the second child. If the parents do not present a bequest rule to the first child, the first child resides at f_0, the location in which the first child maximizes his or her own utility when the bequest distribution ratio is zero[3]. Having observed this, the parents then present the second child a bequest rule that relates the location s to the bequest distribution ratio β_s. That is, the

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[3] f_0 is derived in 4-2.

parents choose s and β_s which maximize their utility subject to the second child's participation constraint:

$$u_k(Y_k(s)+\beta_s b)+v_k(a(f_0)+a(s))-c(a(s)) \geq u_k(Y_k(s_0))+v_k(a(f_0)+a(s_0))-c(a(s_0))(\equiv \bar{U}^s), \tag{1}$$

where s_0 is the location of the second child's that maximizes his or her own utility when the bequest distribution ratio is zero, and $\bar{U}^s$ is the reservation utility level of the second child[4].

3-2 Derivation of s_0

We now examine the level of s_0 in the second child's participation constraint (1). The following assumption is made in deriving s_0.

Assumption 1: $\hat{s} < k^c$

$\hat{s}(=\arg\max_s v_k(a(f_0)+a(s))-c(a(s)))$ is the location that maximizes the second child's utility derived from attention. Namely, $\hat{s}$ satisfies $\left[v_k'(a(f_0)+a(s))-c'(a(s))\right]a'(s)=0$.

Noting that k^c is the location that maximizes the income and that s_0 is determined by the level of utility from attention and the level of utility from income, we have that, if $\hat{s} < k^c$, then s_0 is less than or equal to k^c and the second child does not reside farther than k^c. On the other hand, if $\hat{s} > k^c$, then s_0 is greater than k^c and the second child lives farther than k^c. The former case is assumed here in Assumption 1. That is, k^c is assumed to be far enough from the parents' location for the utility from attention to diminish.

Figure 3 shows the relationship between the second child's location and the second

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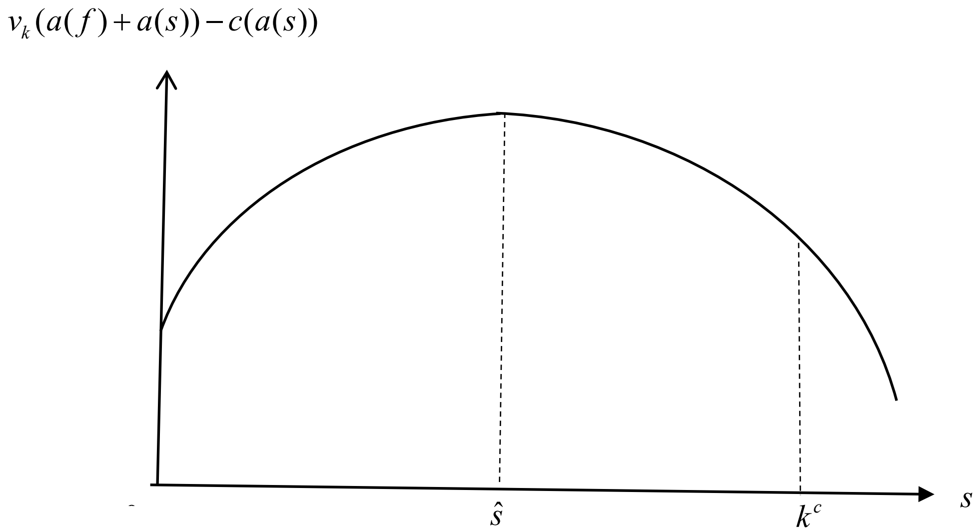
[4] If the second child refuses the bequest rule, the distribution should be made equally between the first and second child according to the law. The reason why the bequest distribution ratio is zero when the second child refuses the bequest rule in (1) is that reducing the level of the second child's reservation utility increases the bargaining power of the parents and allows them to extract a more favorable (s, β_s) for the parents from the second child. The parents do not increase their utility by making the distribution equal, because the bequest distribution ratio does not affect the parents' utility. If the second child rejects the bequest rule, the parents give the entire bequest to the first child.

child's utility from attention in the case of $\hat{s} < k^c$.

Figure 3 The second child's utility from attention



Proposition 1

Under Assumption 1, we have $\hat{s} < s_0 \le k^c$.

(Proof) See Appendix 1.

s_0 is the location that maximizes the sum of the utility obtained from attention and the utility obtained from income. Therefore, when s is smaller than or equal to $\hat{s}$, since both the utility from attention and the utility from income become larger the further away from the parents' location (point 0), s_0 does not exist in this region. When s is larger than $\hat{s}$ and is smaller than or equal to k^c, the utility derived from attention becomes smaller the further away from $\hat{s}$ while the utility derived from income becomes larger the further away from $\hat{s}$ and closer to k^c. When s is larger than k^c, both the utility obtained from attention and the utility from income decrease as the second child moves further away from k^c, so that the second child resides no further than k^c. Thus, s_0 is obtained at a location which is larger than $\hat{s}$ and is smaller than or equal to k^c.

In the following, we consider the case of $s_0 = k^c$. This case would arise when $Y_k'(s)$ is sufficiently large for $s < k^c$.

3-3 Solving the problem for the parents

The parents present the second child a location s and a bequest distribution ratio β_s, which maximize the parents' utility subject to the second child's participation constraint[5]. They are given by the solution to the following problem $(s^*,\ \beta_s^*)$:

$$\begin{aligned} &\underset{s,\,\beta_s}{Max}\, u_p(Y_p - b) + v_p(a(f_0) + a(s)) \\ &\textit{sub to } (1),\ 0 < \beta_s \le 1,\ s \ge 0,\ \ \textit{given } b,\, f_0. \end{aligned} \tag{2}$$

Proposition 2

In the equilibrium for the subgame that begins at P_3, the bequest rule presented to the second child is as follows.

(i) If $s = 0$ does not satisfy the participation constraint for the second child (the second child never lives with the parents), then $s^* > 0,\ \beta_s^* = 1$.

(ii) If $s = 0$ satisfies the participation constraint for the second child, then $s^* = 0,\ \hat{\beta}_s \le \beta_s^* \le 1$,

where $\hat{\beta}_s$ is the level of β_s that satisfies the second child's participation constraint with $s = 0$:

$$u_k(Y_k(0) + \beta_s b) + v_k(a(f_0) + a(0)) - c(a(0)) = \bar{U}_s \tag{3}$$

(Proof) See Appendix 2 and 3

Figure 4 demonstrates Proposition 2 (i). When the second child's participation constraint is satisfied with equality, the indifference curve satisfying the reservation utility level $\bar{U}_s$ is represented by AB. In order to keep the second child's utility level constant, s must be lowered when β_s is raised, so the indifference curve AB has a downward slope to the right[6]. The second child's participation constraint is satisfied in the region

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5 This formulation is according to Futagami *et al*. (2006).

6 Differentiating $u_k(Y_k(s) + \beta_s b) + v_k(a(f_0) + a(s)) - c(a(s)) = \bar{U}^s$ with respect to s and β_s leads to the following equation:

above the indifference curve AB and bounded by $\beta_s = 1$. The parents' indifference curves I_p are vertical because the parents' utility does not depend on β_s. Since $a'(s) < 0$, the parents' utility increases as I_p moves to the left and maximized at A in the feasible region that satisfies the participation constraint of the second child. Proposition 2 (i) is the case where the second child does not live with the parents.

Figure 4 Proposition 2 (i): $s^* > 0$ and $\beta_s^* = 1$

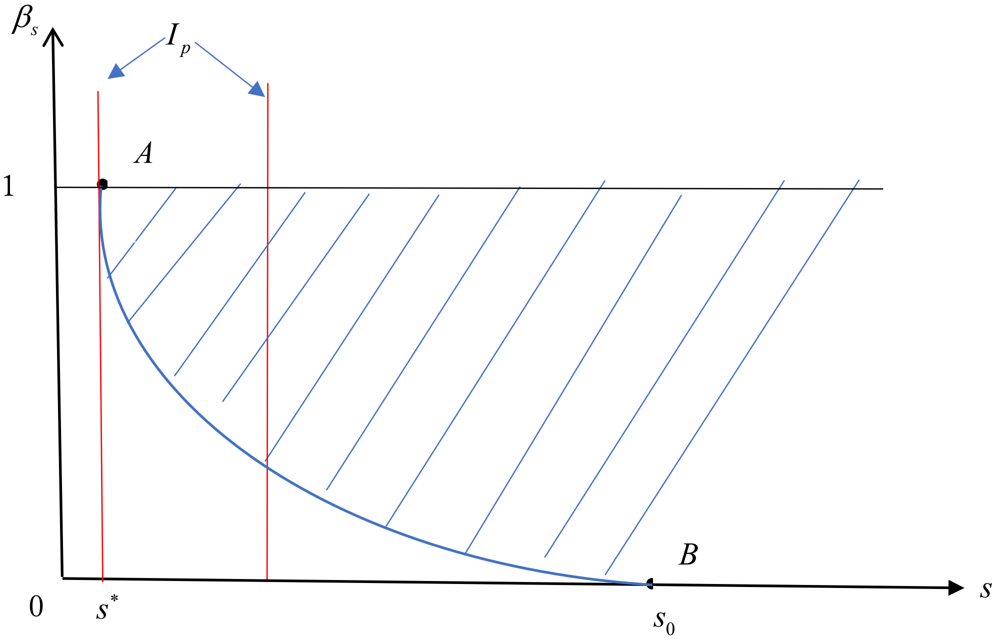


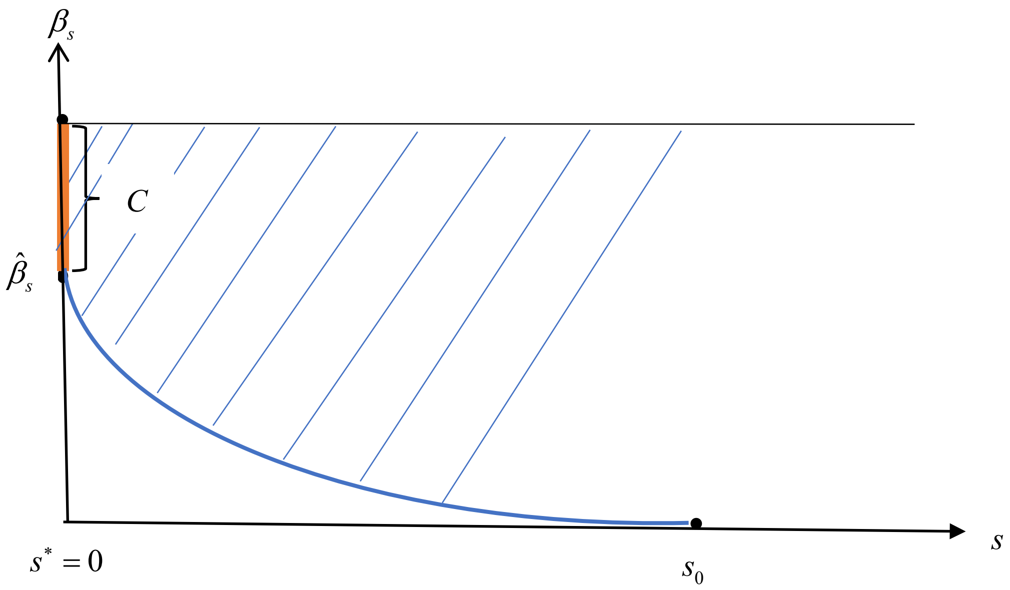
Figure 5 demonstrates Proposition 2 (ii). Since the parents' indifference curves are vertical and the utility level increases as one moves to the left, the segment C: $s^* = 0,\ \hat{\beta}_s \le \beta_s^* \le 1$ maximizes the parents' utility in the region satisfying the second child participation constraint and $\beta_s \le 1$.

$$d\beta_s / ds = -\left\{u_k'(Y_k(s)) + \beta_s b)Y_k'(s) + \left[v_k'(a(f_0) + a(s)) - c'(a(s))\right]a'(s)\right\} / bu_k' .$$

When $s^* > 0$, the sign of the numerator of the above equation is positive from (A3) with equality, $v_p'(a(f_0) + a(s)) > 0$ and $a'(s) < 0$. Thus, we have that $d\beta_s / ds$ is negative.

Proposition 2 (ii) is the case where the second child lives with the parent. Note that the second child accepts to live with the parents even if the bequest distribution ratio is less than 1. The reason why the solution is not unique is as follows. When $\hat{\beta}_s \le \beta_s \le 1$, the second child lives with the parents, while the location of the first child does not change even if β_s changes because the location of the first child is a predetermined variable. Therefore, when $\hat{\beta}_s \le \beta_s^* \le 1$ and $s^* = 0$, both the total attention and the utility level of the parents are constant.

Figure 5 Proposition 2 (ii): $s^* = 0$ **and** $\hat{\beta}_s \le \beta_s^* \le 1$



4. Bequest rule the parents present to the first child

In this section, we examine the bequest rule the parents present to the first child. More specifically, we derive the rule that relates the first child's location f to the bequest distribution ratio β_f $(0 \le \beta_f \le 1)$ in the equilibrium of the subgame after P_2 in Figure 2.

4-1 First child's participation constraint

When the parents present the bequest rule to the first child, the participation constraint is as follows.

$$\begin{aligned} &U_k(f,\ \beta_f,\ b) \equiv u_k(Y_k(f)+\beta_f b)+v_k(a(f)+a(s(f,\ \beta_f,\ b))-c(a(f)) \\ &\geq u_k(Y_k(f_0))+v_k(a(f_0)+a(s^*(f_0,\ b)))-c(a(f_0))(\equiv \bar{U}_f), \end{aligned} \tag{4}$$

where $\bar{U}_f$ is the reservation utility level of the first child. $s(f,\ \beta_f,\ b)$ on the left-hand side of (4) is the reaction function of the second child, and given by

$$s(f,\ \beta_f,\ b)=\begin{cases} \bar{s}(f,\ \beta_f,\ b), & if\ \beta_f>0 \\ s^*(f_0,\ b), & if\ \beta_f=0, \end{cases} \tag{5}$$

where $\bar{s}(f,\ \beta_f,\ b)$ is the location chosen by the second child (at S_1 in Figure 2) after the first child accepts the parents' bequest rule ($\beta_f>0$). Namely[7],

$$\bar{s}(f,\ \beta_f,\ b)=\arg\max_s u_k(Y_k(s)+(1-\beta_f)b)+v_k(a(f)+a(s))-c(a(s)). \tag{6}$$

On the other hand, the bequest rule $f=f_0,\ \beta_f=0$, which is the same as the threat point, satisfies the first child's participation constraint. However, the parents do not present such a bequest rule to the first child, and present the bequest rule $(s^*(f_0,\ b),\ \beta_s^*(f_0,\ b))$ to the second child as shown in Section 3.

Participation constraint for the first child (4) is shown in Figure 6. In Figure 6, $A(f_0,\ 0)$ satisfies the participation constraint as mentioned above.

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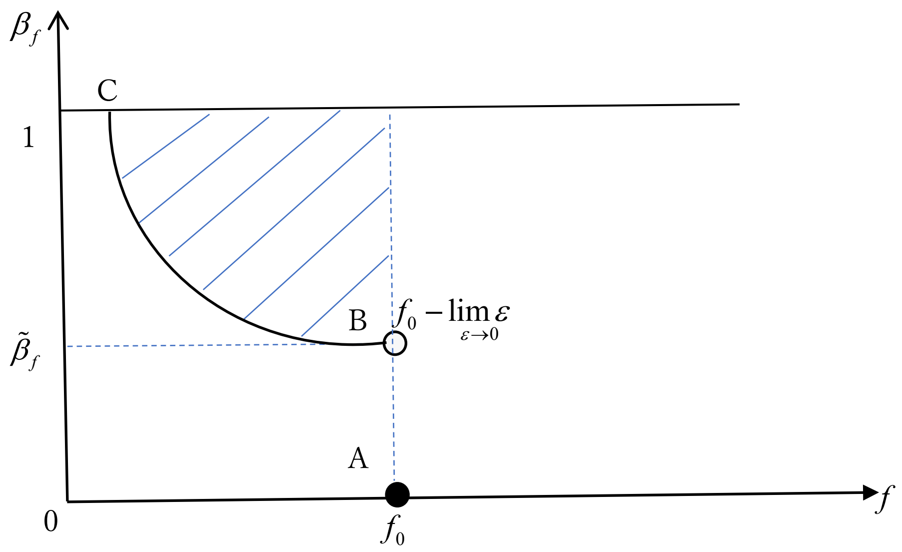
[7] We have $\partial\bar{s}/\partial f<0$, $\partial\bar{s}/\partial\beta_f>0$ and $\partial\bar{s}/\partial b<0$. See Appendix 5 for the derivation.

Figure 6 Participation constraint for the first child



In order to induce the first child to live closer to the parent's location than f_0, the parent must present a positive β_f to the first child. When β_f becomes positive, the reaction function of the second child changes from $s^*(f, b)$ to $\bar{s}(f, \beta_f, b)$. This implies that the second child's location moves away from the parents' location: $s^*(f, b) < \bar{s}(f, \beta_f, b)$ [8]. That is, when f changes marginally from f_0 to $f_0 - \lim_{\varepsilon \to 0} \varepsilon$, s changes discontinuously from s^* to $\bar{s}$ and the level of attention of the second child $a(s(f, \beta_f, b))$ decreases discontinuously. Therefore, to keep the utility level of the first child constant, the bequest distribution ratio to the first child must increase discontinuously because $\partial U_f / \partial \beta_f \big|_{f=f_0} = bu_k'(Y_k(f_0) + \beta_f b) > 0$. That is, there exists $\tilde{\beta}_f$ such that

[8] See Appendix 6 for the proof.

$$u_k(Y_k(f_0 - \lim_{\varepsilon\to 0}\varepsilon) + \lim_{\varepsilon\to 0}\tilde{\beta}_f b) + v_k(a(f_0 - \lim_{\varepsilon\to 0}\varepsilon) + a(\overline{s}(f_0 - \lim_{\varepsilon\to 0}\varepsilon,\ \lim_{\varepsilon\to 0}\tilde{\beta}_f,\ b)) - c(a(f_0 - \lim_{\varepsilon\to 0}\varepsilon))$$
$$= u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\ b))) - c(a(f_0)). \tag{7}$$

In this case, the first child's participation constraint jumps upward from A to B at $f = f_0$. The following lemma proves the existence of $\tilde{\beta}_f$.

Lemma 1 There exists $\tilde{\beta}_f$ that satisfies (7) (that is, $\lim_{\varepsilon\to 0}\tilde{\beta}_f \neq 0$).

(Proof) Substituting $\lim_{\varepsilon\to 0}\varepsilon = 0$ into (7) yields

$$\begin{aligned}&\left\{u_k[Y_k(f_0) + (\lim_{\varepsilon\to 0}\tilde{\beta}_f)b] - u_k[Y_k(f_0)]\right\}\\&+\left\{v_k[a(f_0) + a(\overline{s}(f_0,\ \lim_{\varepsilon\to 0}\tilde{\beta}_f,\ b)] - v_k[a(f_0) + a(s^*(f_0,\ b))]\right\}\\&-\left\{c[a(f_0)] - c[a(f_0)]\right\} = 0.\end{aligned} \tag{8}$$

Applying the mean value theorem to $u_k(\cdot)$, $v_k(\cdot)$ and $a(\cdot)$ in (8) yields

$$u_k'(\gamma)(\lim_{\varepsilon\to 0}\tilde{\beta}_f)b + v_k'(\delta)a'(\rho)[\overline{s}(f_0,\ \lim_{\varepsilon\to 0}\tilde{\beta}_f,\ b) - s^*(f_0,\ b)] = 0, \tag{9}$$

where

$$\gamma \in [Y_k(f_0),\ Y_k(f_0) + (\lim_{\varepsilon\to 0}\tilde{\beta}_f)b],\ \delta \in [a(f_0) + a(\overline{s}(f_0,\ \lim_{\varepsilon\to 0}\tilde{\beta}_f,\ b)),\ a(f_0) + a(s^*(f_0,\ b))]$$

and $\rho \in [s^*(f_0,\ b),\ \overline{s}(f_0,\ \lim_{\varepsilon\to 0}\tilde{\beta}_f,\ b)]$.

Suppose that $\lim_{\varepsilon\to 0}\tilde{\beta}_f = 0$. In this case, the first term of (9) is zero. However, since $s^*(f_0,\ b) < \overline{s}(f_0,\ 0,\ b)$, (9) is no longer valid. This implies that the contradiction arises. Therefore, we have $\lim_{\varepsilon\to 0}\tilde{\beta}_f \neq 0$. □

When $f < f_0,\ \beta_f > \tilde{\beta}_f$, we have $s(f,\ \beta_f,\ b) = \overline{s}(f,\ \beta_f,\ b)$. In this case, the participation constraint for the first child is

$$u_k(Y_k(f)+\beta_f b)+v_k(a(f)+a(\overline{s}(f,\ \beta_f,\ b))-c(a(f))$$
$$\geq u_k(Y_k(f_0))+v_k(a(f_0)+a(s^*(f_0,\ b)))-c(a(f_0)),\qquad f\in[0,\ f_0). \tag{10}$$

In Figure 6, (10) corresponds to BC and the region above it[9].

When the participation constraint (10) is satisfied with equality, it is illustrated as BC satisfying the reservation utility level $\overline{U}_f$ in Figure 6. When β_f is raised, f must be lowered in order to keep the utility level of the first child constant, so the slope of BC is negative. Regarding the location of point C, we have two cases: one is where BC intersects with $\beta_f = 1$ and the other is where BC intersects with the vertical axis ($f = 0$). Figure 6 shows the former case.

4-2 Derivation of f_0

We now examine the level of f_0 in the first child's participation constraint . The following assumption is made in deriving f_0.

Assumption 2: $\hat{f} < k^c$

$\hat{f}\,(=\arg\max_f v_k(a(f)+a(s^*(f,\ b)))-c(a(f))\,)$ is the location that maximizes the first child's utility derived from attention. Namely, $\hat{f}$ satisfies $\left[v_k'(a(f)+a(s^*(f,\ b)))\right]\left[a'(f)+a'(s^*)s_f^*\right]-c'\cdot a'(f)=0$ [10]. Noting that k^c is the

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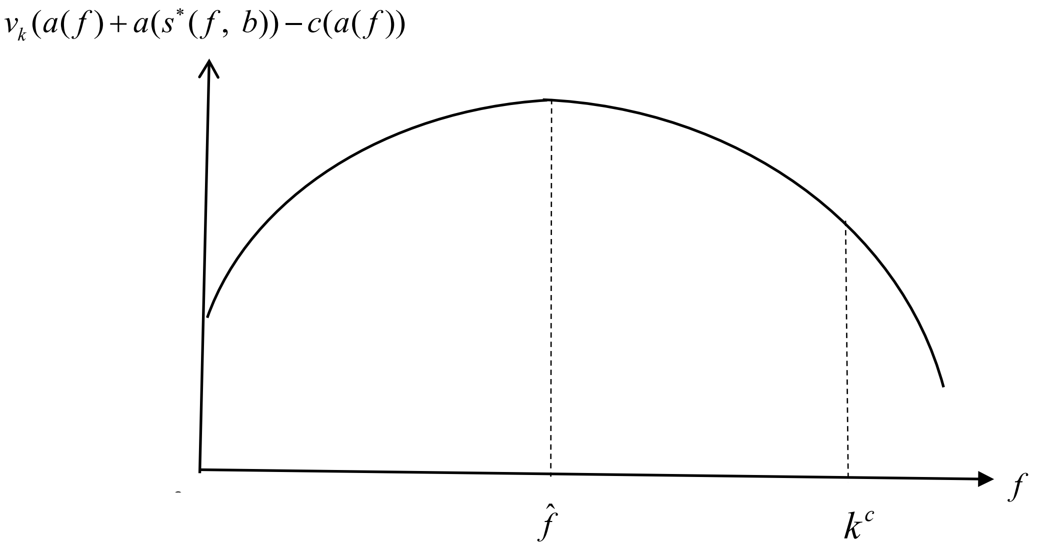
[9] A bequest rule with $\beta_f > 0$ and $f > f_0$ also satisfies the first child's participation constraint, but such a bequest rule never arises in the equilibrium. Thus, we ignore the case of $f > f_0$.

[10] For $\hat{f}$ to be an interior solution, $a'(\hat{f})+a'(s^*)s_{\hat{f}}^* < 0$ must hold. While the sign of $a'(f)+a'(s^*)s_f^*$ is indeterminate, we assume it to be negative. $a'(f)+a'(s^*)s_f^*$ represents the change in the total attention to the parents when f increases. It can be divided into the direct effect ($a'(f) < 0$) and the indirect effect ($a'(s^*)s_f^* > 0$). The direct effect is that an increase in f decreases the attention to the parents and the indirect effect is that increase in f leads to a decrease in s^* which increases the attention to the parents. We assume that the former effect is larger than the latter. See Appendix 7 for details.

location maximizing the children's income and that f_0 is determined by the level of utility from attention as well as the level of utility from income, we have that, if $\hat{f} < k^c$, then f_0 is less than or equal to k^c and the first child does not reside farther than k^c. On the other hand, if $\hat{f} > k^c$, then f_0 is greater than k^c and the first child lives farther than k^c. The former case is assumed here in Assumption 2. That is, k^c is assumed to be far enough from the parents' location for the utility from attention to diminish.

Figure 7 shows the relationship between the first child's location and the first child's utility from attention in the case of $\hat{f} < k^c$.

Figure 7 The first child's utility from attention



Proposition 3

Under Assumption 2, we have $\hat{f} < f_0 \leq k^c$.

(Proof) See Appendix 8.

f_0 is the location that maximizes the sum of the utility obtained from attention and the utility obtained from income. Therefore, when f is smaller than or equal to $\hat{f}$, both the utility from attention and the utility from income become larger the further away from the parents' location, implying that f_0 does not exist in this region. When f is larger than $\hat{f}$ and smaller than or equal to k^c, the utility derived from attention becomes smaller while the utility derived from income becomes larger the further away from $\hat{f}$ and closer to k^c. When f is larger than k^c, both the utility obtained from attention and the utility from income decrease as the first child moves further away from k^c, so that the first child resides no further than k^c. Thus, the sum of the utility is maximized and f_0 is obtained at a location which is larger than $\hat{f}$ and smaller than or equal to k^c.

In the following, we consider the case of $f_0 = k^c$. This case would arise when $Y_k'(f)$ is sufficiently large for $f < k^c$.

4-3 Solving the problem for the parents

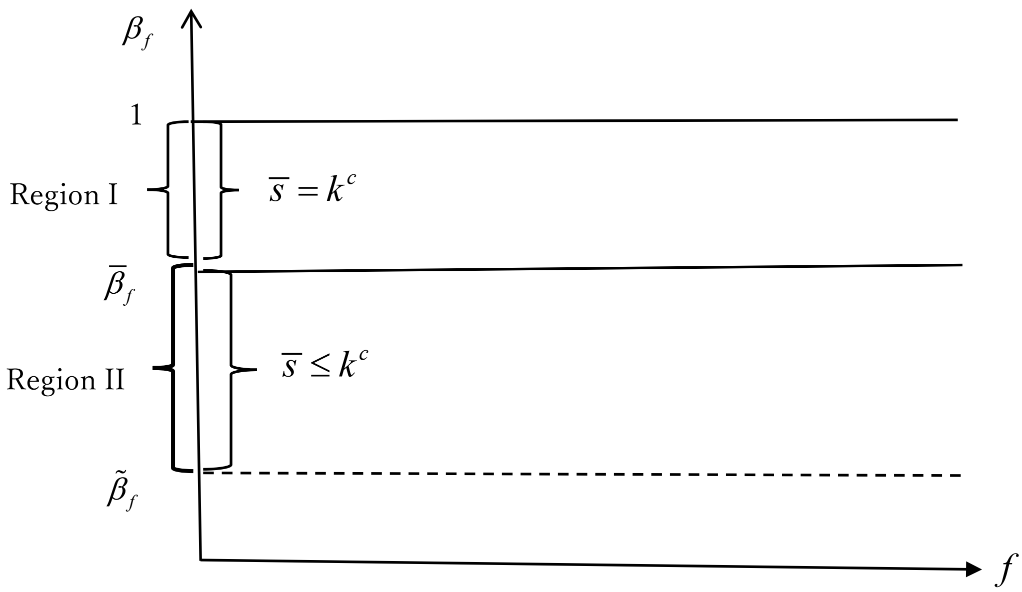
In this subsection, we derive the first child's location f and the bequest distribution ratio β_f that the parents present to the first child. To do this, we first examine $\bar{s}$ in the first child's participation constraint (BC in Figure 6).

If $\beta_f = 1$, the bequest given to the second child is 0, implying that $\bar{s} = s_0 = k^c$. If $\beta_f < 1$, the bequest given to the second child is $(1-\beta_f)b$. In this case, the second child resides either at k^c (corner solution) or at $s \le k^c$ (interior solution). Which of the two is the case depends on how much $(1-\beta_f)b$ is. That is, if $(1-\beta_f)b$ is small enough, the second child resides at k^c. But, since the marginal utility of consumptions for the second child decreases as $(1-\beta_f)b$ increases, there must be a threshold at which the second child's location switches to the interior solution. The threshold for such a bequest distribution ratio to the first child is defined as $\bar{\beta}_f$ [11].

[11] We define $\bar{\beta}_f$ as β_f that satisfies the FOC of the second child when $\bar{s} = k^c$:

We now define Regions I and II as in Figure 8.

Figure 8: Region I and Region II



Region I has β_f with $\overline{\beta}_f < \beta_f \le 1$ and Region II has β_f with $\tilde{\beta}_f < \beta_f \le \overline{\beta}_f$. We have $\overline{s} = k^c$ (corner solution) in Region I and $\overline{s} \le k^c$ (interior solution) in Region II[12].

We now present Lemma 2 and Assumption 3 regarding the relative magnitude between the marginal rate of substitution for the first child (MRS^K) and the marginal rate of substitution for the parents (MRS^P).

Lemma 2

In Region I, $\left|MRS^K\right| < \left|MRS^P\right|$ ($MRS^K > MRS^P$),

where

$MRS^P = -U_f^P / U_{\beta_f}^P < 0$,

$MRS^K = -U_f^K / U_{\beta_f}^K < 0$,

$$\partial U_k / \partial \overline{s}\big|_{\overline{s}=k^c} = u_k'(Y_k(k^c) + (1-\beta_f)b) \cdot Y_{k-}'(k^c) + [v_k'(a(f) + a(k^c)) - c'(a(k^c))]a'(k^c) = 0.$$

12 A detailed explanation of Region I and Region II is provided in Appendix 9.

$$U_f^k = u_k'(Y_k(f) + \beta_f b)Y_k'(f) + v_k'(a(f) + a(\overline{s}(f,\ \beta_f,\ b))) \cdot \left(a'(f) + a'(\overline{s})\overline{s}_f\right) - c' \cdot a'(f)) > 0,$$

$$U_{\beta_f}^P = v_p'(a(f) + a(\overline{s})) \cdot a'(\overline{s})\overline{s}_{\beta_f} < 0,$$

$$U_{\beta_f}^K = bu_k'(Y_k(f) + \beta_f b) + v_k'(a(f) + a(\overline{s}(f,\ \beta_f,\ b))) \cdot a'(\overline{s})\overline{s}_{\beta_f} > 0.$$

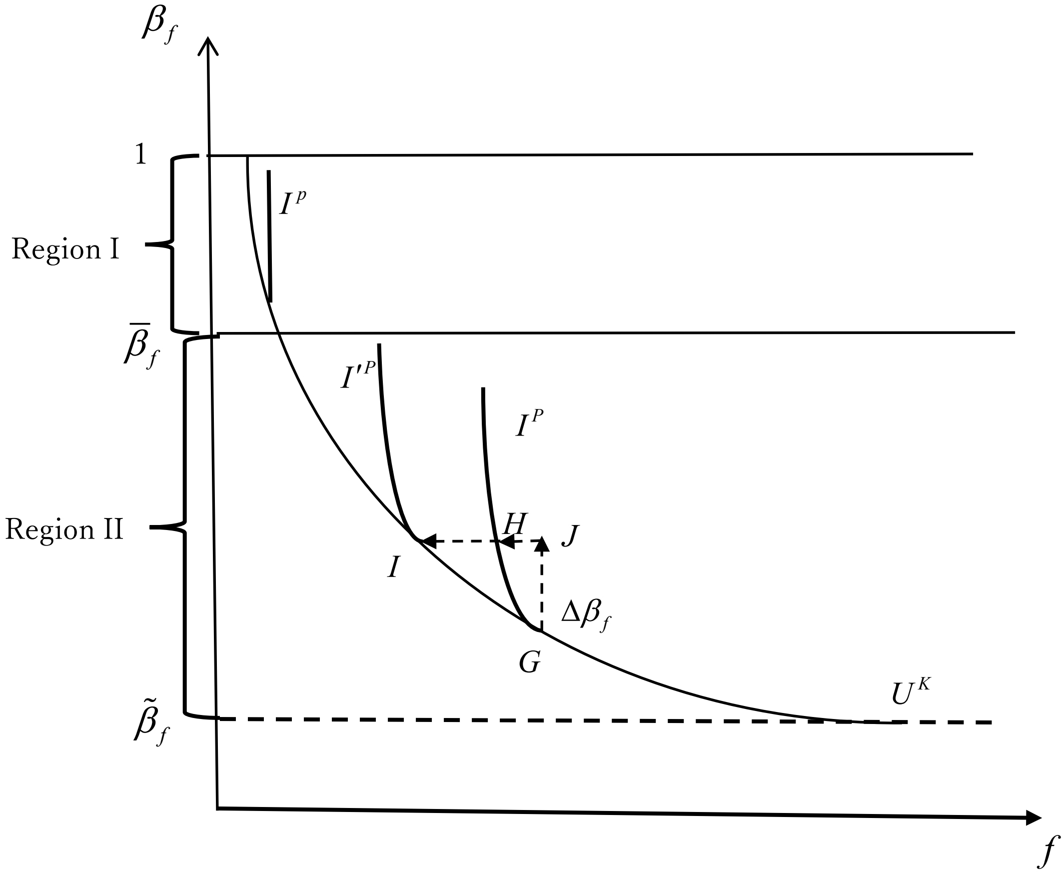
The utility function of the parents in Region I is $U^P = u_p(Y_p - b) + v_p(a(f) + a(k^c))$. The parents' indifference curve I^P in Figure 9 is vertical because the parents' utility does not depend on β_f. Therefore, we have $\left|MRS^K\right| < \left|MRS^P\right|$.

The utility function of the parents in Region II is $U^P = u_p(Y_p - b) + v_p(a(f) + a(\overline{s}(\beta_f, f))$, so the slope of the parents' indifference curve I^P is negative. In Region II, we make the following assumption.

Assumption 3

In Region II, $\left|MRS^K\right| < \left|MRS^P\right|$ $(MRS^K > MRS^P)$

Figure 9 Parents' indifference curves on the first child's participation constraint



We now discuss the implications and validity of Assumption 3. When β_f increases by $\Delta\beta_f$, the level of the parents' utility decreases because the second child moves away from the parents and the second child's attention decreases as the bequest given to the second child $(1-\beta_f)b$ decreases. How much the first child's location f needs to move closer to the parents' to increase the level of the first child's attention in order to maintain the same parental utility level is indicated by $1/\left|MRS^P\right|$. In contrast, when β_f increases by $\Delta\beta_f$, the utility level of the first child rises because the bequest given to the first child $\beta_f b$ increases. How much f needs to move closer to the parents' to maintain the same level of utility for the first child is indicated by $1/\left|MRS^K\right|$ (when the first child moves closer to the parents, the utility of the first child decreases due to a decrease in income and other factors). From $d^2a(k)/dk^2 > 0$ $(k = f,\ s)$, the change in attention due to a marginal change in the child's location becomes smaller as the child's location moves away from the parents'. When the second child resides farther from and the first child resides closer to the parents' location, the decrease in attention due to a marginal increase in s is much smaller than the decrease in attention due to a marginal increase in f. Thus, when β_f increases, a slight decrease in f can maintain the utility level of the parents. This means that $\overline{JH}$ is very small in Figure 9, and thus it is reasonable to assume that $\overline{JH} < \overline{JI}$ holds, namely, to make Assumption 3[13].

Lemma 3

In both Region I and Region II, the utility level of the parents is higher in the upper left on the first child's participation constraint in Figure 9.

We now explain Lemma 3 using Figure 9. In Region I, since the attention to the parents increases as the first child's location f is smaller, the utility level of the parents is higher in the upper left on the participation constraint of the first child.

In Region II, when β_f increases by $\Delta\beta_f$ at G, since the marginal rate of substitution for the parents is greater than that for the first child, the first child needs to be closer to the parent up to H to maintain the parents' utility level, whereas the first child needs to be closer to the parent up to I (closer to the parents' location than H) to maintain the first child's utility level. Therefore, when comparing the utility level of the parents at G (or H) on I^p and at I on $I^{p\prime}$, the latter is higher than the former. This is shown mathematically below.

Totally differentiating the parents' utility functions $U^p = U^p(f,\ \beta_f)$ and the first child's participation constraint $U^k(f,\beta_f) = \bar{U}^k$ yield

$$dU^p = U_f^p df + U_{\beta_f}^p d\beta_f = U_{\beta_f}^p[(U_f^p / U_{\beta_f}^p) + (d\beta_f / df)]df\ , \tag{11}$$

13 The details on the conditions for Assumption 3 to hold are shown in Appendix 10.

$$U_f^k df + U_{\beta_f}^k d\beta_f = 0 . \quad (12)$$

Substituting (12) into (11), we have

$$\frac{dU^p}{df} = U_{\beta_f}^p \left[\frac{U_f^p}{U_{\beta_f}^p} - \frac{U_f^k}{U_{\beta_f}^k} \right] = U_{\beta_f}^p \left[-MRS^p + MRS^k \right] . \quad (13)$$

Since $-MRS^p + MRS^k > 0$ from Assumption 3 and $U_{\beta_f}^p < 0$, (13) is negative. Thus, if $df < 0$, then $dU^p > 0$. The level of the parents' utility increases as f moves to the upper left on the first child's participation constraint.

Lemma 3 leads us to Proposition 4.

Proposition 4

In the subgame starting at P_2, the equilibrium solution for the bequest rule the parents present to the first child is as follows.

(i) If $f = 0$ does not satisfy the first child's participation constraint (the first child never lives with the parents), then $f^* > 0$ and $\beta_f^* = 1$.

(ii) If $f = 0$ satisfies the first child participation constraint and $\hat{\beta}_f > \overline{\beta}_f$, then $f^* = 0$ and $\hat{\beta}_f \le \beta_f^* \le 1$.

(iii) If $f = 0$ satisfies the first child participation constraint and $\hat{\beta}_f \le \overline{\beta}_f$, then $f^* = 0$ and $\beta_f^* = \hat{\beta}_f$.

$\hat{\beta}_f$ is β_f that satisfies the participation constraint for the first child $u_p(Y_k(f) + \beta_f b) + v_k(a(f) + a(\overline{s})) - c(a(f)) = \overline{U}_f$ with $f = 0$.

Figure 10 Illustration of the equilibrium: Case (i)

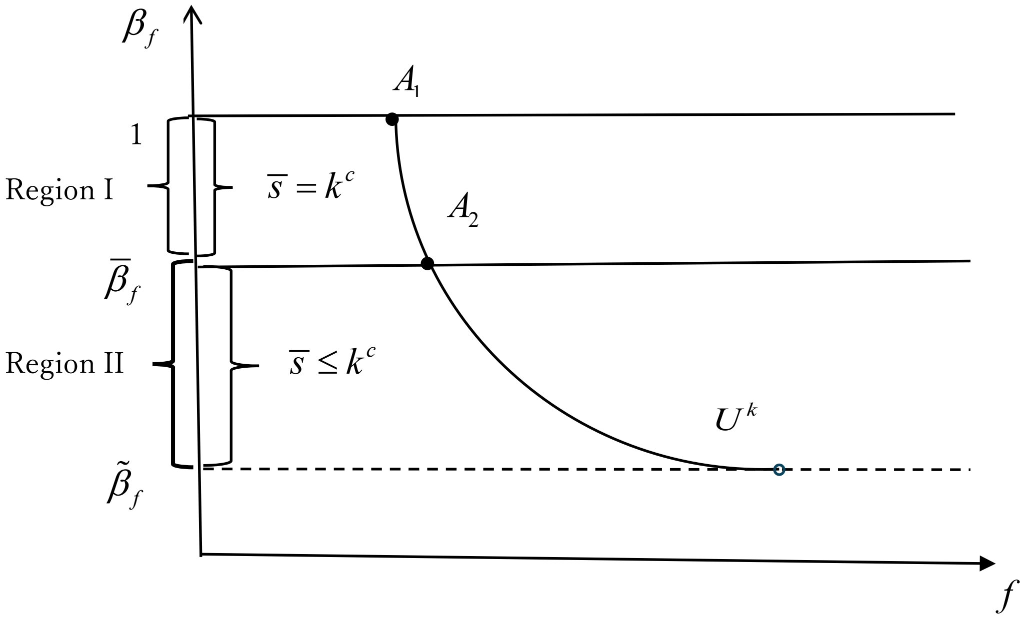


Figure 11 Illustration of the equilibrium: Case (ii)

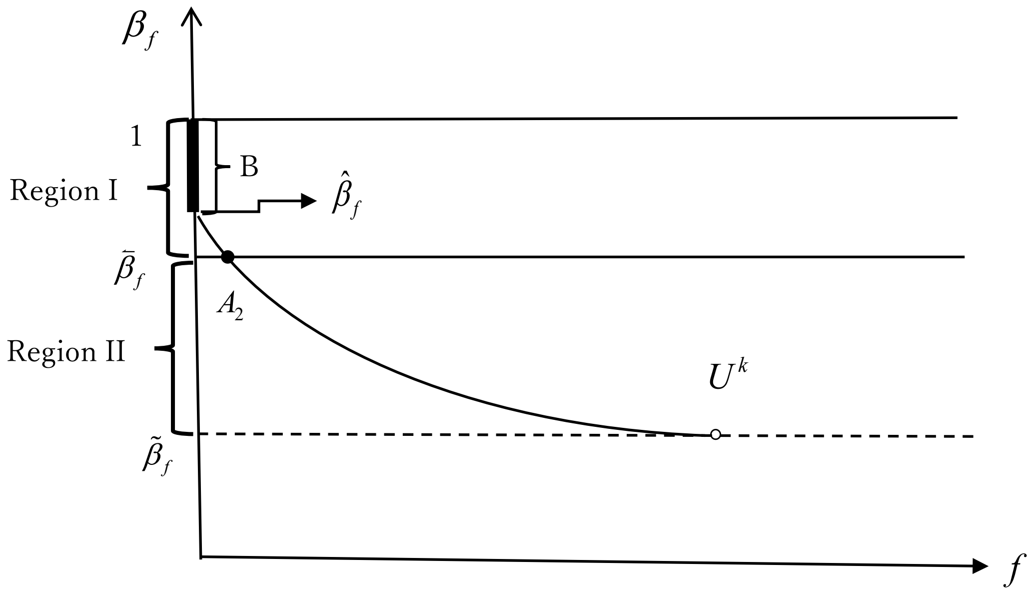
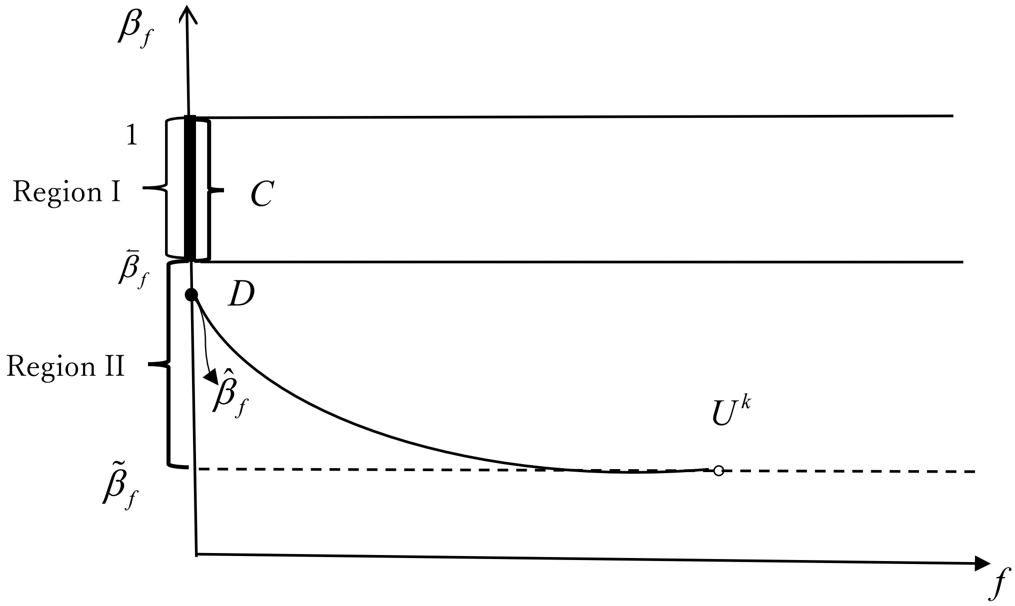


Figure 12 Illustration of the equilibrium: Case (iii)



The intuitive proof of Proposition 4 is as follows[14]. In case (i), since the utility level of the parents is higher in the upper left on the first child's participation constraint, the maximum utility point of the parents is A_1: $f^* > 0,\ \beta_f^* = 1$ in Region I and A_2: $f^* > 0,\ \beta_f^* = \bar{\beta}_f$ in Region II in Figure 10.

We now examine which the parents receive a larger total attention at A_1 or at A_2. At both A_1 and A_2, the location of the second child is k^c. Thus, the total attention is $a(f^*|_{\beta_f^*=1}) + a(k^c)$ at A_1, while it is $a(f^*|_{\beta_f^*=\bar{\beta}_f}) + a(k^c)$ at A_2. Since the slope of the first child's participation constraint is negative, implying that $f^*|_{\beta_f^*=1} < f^*|_{\beta_f^*=\bar{\beta}_f}$, the total attention at A_1 is larger than that at A_2. Thus, the parents' utility is greater at A_1 than at A_2. Therefore, the parents present the bequest rule $f^* > 0,\ \beta_f^* = 1$ to the first child.

[14] The maximum utility for the parents in each region is shown in Appendix 12.

In case (ii), the parents' utility is maximized on B: $f^* = 0,\ \hat{\beta}_f \le \beta_f^* \le 1$ in Region I, because the utility level of the parents is higher in the upper left on the first child's participation constraint in Figure 11. The equilibrium solution is not unique for the following reasons. While the first child lives with the parents if $\beta_f \ge \hat{\beta}_f$, the location of the second child remains k^c in Region I. Therefore, the total attention and the utility level of the parents do not change in $\hat{\beta}_f \le \beta_f^* \le 1$ with $f^* = 0$. The parents' utility maximum point in Region II is A_2: $f^* > 0,\ \beta_f^* = \overline{\beta}_f$. The location of the second child is k^c both on *B* and at A_2. On the other hand, while the first child lives with the parents $(f^* = 0)$ on *B*, the first child lives apart from the parents $(f^* > 0)$ at A_2. Thus, the total attention the parents receive is greater on *B* than at A_2. Therefore, the parents present the bequest rule $f^* = 0,\ \hat{\beta}_f \le \beta_f^* \le 1$ to the first child.

In case (iii), the parents' utility is maximized on *C*: $f^* = 0,\ \overline{\beta}_f \le \beta_f^* \le 1$ in Region I in Figure 12. The maximum utility point of the parents in Region II is D: $f^* = 0,\ \beta_f^* = \hat{\beta}_f$ because the utility level of the parents is higher in the upper left on the first child's participation constraint. The first child's location is $f^* = 0$ both on *C* and at *D*. The second child's location is k^c on C, whereas it is $\overline{s} \le k^c$ at D. Thus, the total attention is $a(0) + a(k^c)$ on C and $a(0) + a(\overline{s})$ at D, implying that the latter is greater than or equal to the former. Therefore, the parents present the bequest rule $f^* = 0,\ \beta_f^* = \hat{\beta}_f$ to the first child.

5. Whether the bequest rule should be presented to the first child or the second child depending on the parents' bequest level

In this section, we examine whether the parents present a bequest rule to the first child or the second child, given the level of the parents' total bequest b. For this purpose, we present the following lemmas which show the relationship between b and the location of each child.

Lemma 4 (relationship between the bequest and the location of the second child)

At P_3 in Figure 2, as b increases, the second child's location s^* in the bequest rule becomes closer to the parents' location 0 ($(ds^* / db)_{\beta_s^*=1} < 0$). When b reaches $\hat{b}_s^0$, the second child lives with the parents, where $\hat{b}_s^0$ is b that satisfies the participation constraint for the second child with $s^* = 0$ and $\beta_s^* = 1$: $U^s(s^*, \beta_s^*, b)\big|_{s^*=0,\, \beta_s^*=1} = \bar{U}^s$.

(Proof) See Appendix 4.

Lemma 5 (relationship between the bequest and the first child's location)

At P_2, as b increases, the first child's location f^* in the bequest rule becomes closer to the parents' location 0 ($(df^* / db)_{\beta_f^*=1} < 0$). When b reaches $\hat{b}_f^0$, the first child lives with the parent, where $\hat{b}_f^0$ is b that satisfies the participation constraint for the first child with $f^* = 0$ and $\beta_f^* = 1$: $U^f(f^*, \beta_f^*, b)\big|_{f^*=0,\, \beta_f^*=1} = \bar{U}^f$.

(Proof) See Appendix 13.

Furthermore, we present the following lemma regarding the relative magnitude between f^* and s^*.

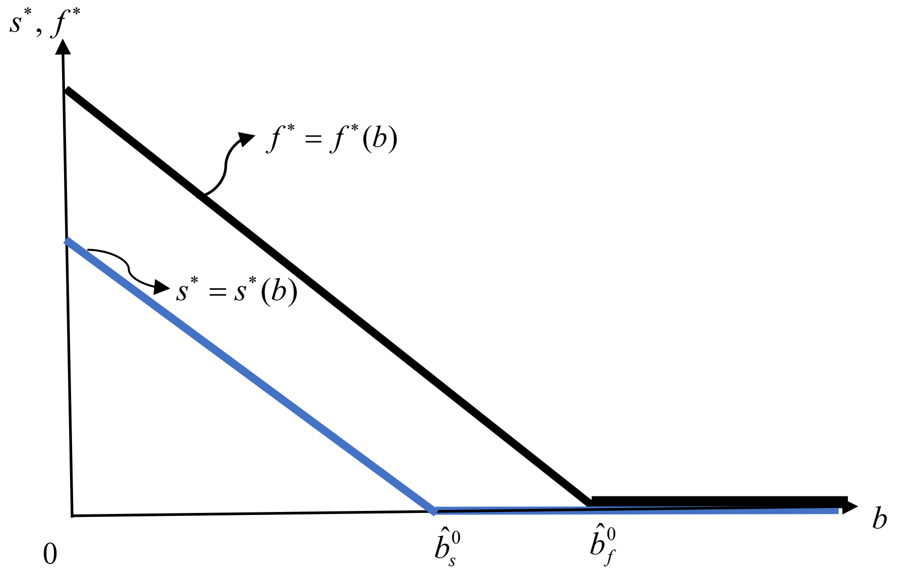
Lemma 6 (relative magnitude between the location of the first child and the location of the second child presented by the parents)

Given $b(>0)$, we have $0 \le s^* < f^*$ for $b < \hat{b}_f^0$ and $s^* = f^* = 0$ for $b \ge \hat{b}_f^0$.

(Proof) See Appendix 14.

Figure 13 shows the relationship between b and s^*, and that between b and f^*.

Figure 13 Total bequest of parents and locations of first and second child



As shown in Lemma 4 and Lemma 5, the more the total bequest is, the closer to the parents' location that of each child presented by the parents is. Furthermore, for $b < \hat{b}_f^0$, the second child's location s^* is closer to the parents' location than the first child's location f^* (Lemma 6). This is because the reservation utility level of the second child's participation constraint is smaller than that of the first child. The reservation utility level of the first child is the utility level when the parents do not present the bequest rule to the first child and the first child resides at k^c, and then the parents present the bequest rule to the second child and the second child resides at s^*. The total attention is then $a(k^c) + a(s^*)$. On the other hand, the reservation utility level of the second child is the utility level when the first child has already resided at k^c (the parents' decision on the bequest rule for the second child comes later than that for the first child), and the parents do not present bequest rule to the second child and second child resides at k^c. The total attention is then $2a(k^c)$. Thus, the total attention at the first child's reservation utility level is larger than that at the second child's reservation utility level. The consumption is $Y_k(k^c)$ both at the first child's and second child's reservation utility levels. Therefore, the first child's reservation utility level is higher than that of the second child[15]. This implies that, for the same level of total bequest $(b < \hat{b}_f^0)$, the parents can induce the second child to live closer to the parents' location than the first child.

Based on the above, the parents decide whether to present the bequest rule to the first child or to the second child. Specifically, comparing the total attention when parents present the bequest rule to the first child with that when they present the bequest rule to the second child, the parents make the decision, given b. We consider three cases depending on the level of b.

(i) $0 < b < \hat{b}_f^0$

When the parents present the bequest rule to the first child, the first child's location is f^* and the second child's location is k^c because the second child is given no bequest ($\beta_f^* = 1$). Thus, the total attention is $a(f^*) + a(k^c)$. On the other hand, when the parents present the bequest rule to the second child, the second child's location is s^* and the first child's location is k^c because the first child was not presented a bequest rule and already resides at k^c. Thus, the total bequest is $a(k^c) + a(s^*)$. From $s^* < f^*$, we have $a(f^*) + a(k^c) < a(s^*) + a(k^c)$. That is, the total attention is greater when the bequest rule is presented to the second child than when presented to the first child. Therefore, the parents present the bequest rule to the second child.

We next examine the case in which b is greater than $\hat{b}_f^0$. For this purpose, we present the following lemma.

---

[15]From $s_0 = f_0 = k^c$, the reservation utility level in the second child's participation constraint is

$$\begin{aligned}\bar{U}_s &\equiv u_k(Y_k(s_0)) + v_k(a(f_0) + a(s_0)) - c(a(s_0)) \\ &= u_k(Y_k(k^c)) + v_k(a(k^c) + a(k^c)) - c(a(k^c)).\end{aligned}$$

On the other hand, the reservation utility level in the first child's participation constraint is

$$\begin{aligned}\bar{U}_f &\equiv u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\ b))) - c(a(f_0)) \\ &= u_k(Y_k(k^c)) + v_k(a(k^c) + a(s^*(k^c,\ b))) - c(a(k^c)).\end{aligned}$$

Since $s^*(k^c,\ b) < k^c$, we have $\bar{U}_s < \bar{U}_f$.

Lemma 7

$$\left.\frac{d\hat{\beta}_f^*}{db}\right|_{f^*=0} < 0,$$

where $\hat{\beta}_f^*$ is β_f^* satisfying $U^f(f^*,\ \beta_f^*,\ b) = \bar{U}_f$ with $f^* = 0$.

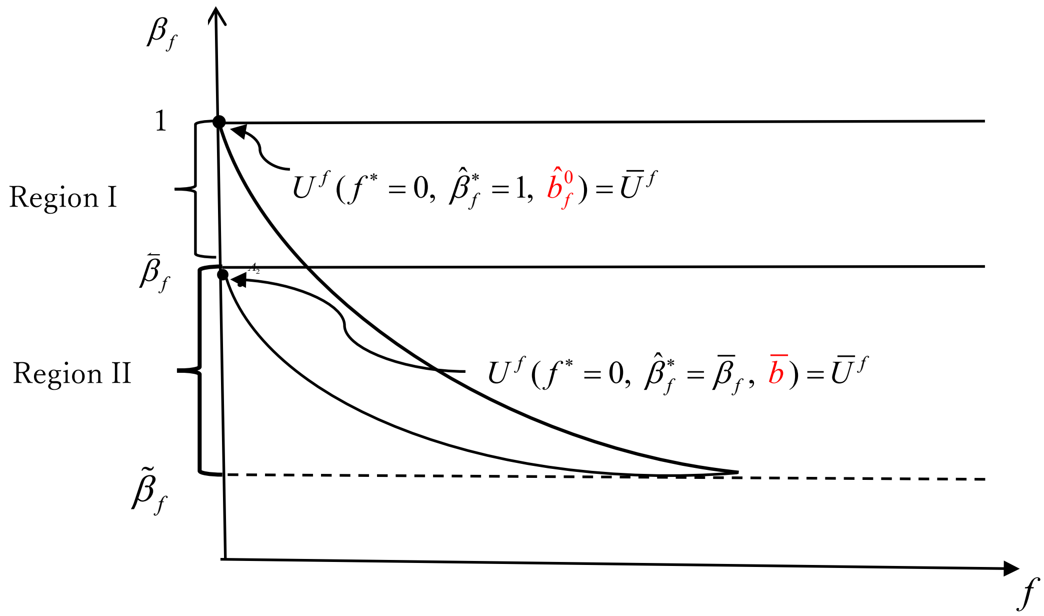
(Proof) See Appendix 15.

In Figure 14, when $b = \hat{b}_f^0$, the first child's participation constraint $U^f(f^*,\ \hat{\beta}_f^*,\ b)\big|_{f^*=0} = \bar{U}^f$ passes through $(f,\ \beta_f) = (0,\ 1)$. Lemma 7 implies that the participation constraint with $f^* = 0$ shifts downward as b increases[16]. Thus, when b increases beyond $\hat{b}_f^0$, we reach $\hat{\beta}_f^* = \bar{\beta}_f$. We define $\bar{b}$ as b satisfying $U^f(f^*,\ \hat{\beta}_f^*,\ b)\big|_{f^*=0,\ \hat{\beta}_f^*=\bar{\beta}_f} = \bar{U}^f$.

---

[16] Lemma 7 can be interpreted as follows. $\hat{\beta}_f^*$ satisfies the first child's participation constraint. In Region I, since the total attention $a(0) + a(k^c)$ and thus the reservation utility level is constant, the bequest given to the first child $\hat{\beta}_f^* b$ must be also constant. Thus, an increase in the total bequest b results in a decrease in $\hat{\beta}_f^*$. In Region II, the total attention increases as the second child's location $\bar{s}$ moves closer to the location of the parents from $\bar{s}_b < 0$. Since the reservation utility level is constant, the bequest given to the first child $\hat{\beta}_f^* b$ will decrease. In Region II, an increase in b lowers $\hat{\beta}_f^*$ further, compared to Region I.

Figure 14 Total bequest of parents and the first child's participation constraint



(ii) $\hat{b}_f^0 \le b \le \bar{b}$

When the parents present the bequest rule to the first child, the first child's location is $f^*=0$ and the second child's location is k^c because the second child is given some bequest but still resides at k^c in Region I. Thus, the total attention is $a(0)+a(k^c)$. On the other hand, when the parents present the bequest rule to the second child, the second child's location is $s^*=0$, and the first child's location is k^c because the first child was not presented a bequest rule and already resides at k^c. Thus, the total bequest is $a(k^c)+a(0)$. The total attention is the same whether the parents present the bequest rule to the first child or to the second child. Therefore, the parents' utility is indifferent whether they present the bequest rule to the first child or to the second child.

(iii) $b > \bar{b}$

When the parents present the bequest rule to the first child, the first child's location is $f^*=0$ and the second child's location is $\bar{s}$ because the second child, who is given more bequest than in case (ii), resides at $\bar{s}(<k^c)$. Thus, the total attention is $a(0)+a(\bar{s})$. On the other hand, when the parents present the bequest rule to the second child, the second child's location is $s^*=0$, and the first child's location is k^c because the first child was not presented a bequest rule and already resides at k^c. Thus, the total bequest is $a(k^c)+a(0)$. From $\bar{s}<k^c$, we have $a(0)+a(\bar{s})>a(k^c)+a(0)$. Therefore, the parents present the bequest rule to the first child.

The above analysis is summarized in Proposition 5 below.

Proposition 5

(i) If $0 < b < \hat{b}_f^0$, then the parents present a bequest rule to the second child and the second child lives with the parents or closer to the parents than the first child. The first child resides at k^c.

(ii) If $\hat{b}_f^0 \le b \le \overline{b}$, the parents present a bequest rule to either the first child or the second child. One who is presented a bequest rule lives with the parents and the other resides at k^c.

(iii) If $b > \overline{b}$, the parents present a bequest rule to the first child, and the first child lives with the parents. The second child resides closer to the parents' location than k^c.

Proposition 5 suggests that the location pattern of siblings differs depending on the total amount of parental bequests.

When the total amount of bequests is less than $\hat{b}_f^0$, the parents present a bequest rule to the second child, and the second child lives with or closer to the parents than the first child. Since the reservation utility of the second child is lower than that of the first child, the parents can induce the second child to reside closer to their location than the first child under a same level of b. The reason why the second child's level of reservation utility is lower is that the stage in which the parents present the bequest rule to the first child is earlier than the stage in which the parents present it to the second child. This implies that the first child has already resided at k^c when the parents present the bequest rule to the second child (birth-order effect).

When the total amount of bequests exceeds $\hat{b}_f^0$, presenting a bequest rule to the first child and presenting it to the second child become indifferent for the parents. The first or second child lives with the parents, and the one who does not live with them lives at k^c. This is because, in this range of b, either of the children accepts the bequest rule that requires cohabitation with the parents.

As the total amount of bequests increases further and exceeds $\overline{b}$, the parents present a bequest rule to the first child. The first child lives with the parents and the second child lives far away but closer to the parents' location than k^c. This is because the total attention is greater when the bequest rule is presented to the first child than when the bequest rule is presented to the second child. Even when the bequest rule, which requires living with the parents, is presented to the first child, a portion of bequests is given to the second child and the second child's disposable income increases, leading to the second child living closer to the parents' location than k^c. On the other hand, when the bequest rule, which requires living with the parents, is presented to the second child, the first child has already lived at k^c because the location choice of the first child is earlier than that of the second child (birth-order effect).

6. Conclusion

Using a strategic bequest model, in which the parents present a bequest rule that relates the child's place of residence to the bequest distribution ratio, this paper examines how the distribution of bequests among children and the location of each child are determined.

The main result obtained in this paper is that family location pattern and bequest distribution among children differ depending on the total amount of parental bequests. When the total amount of bequests is less than a certain level, the parents present a bequest rule to the second child, and the second child lives closer to the parents than the first child (the second child may live with the parents). In this case, the second child receives a larger part (or all) of the parental bequests. When the total amount of bequests exceeds that level, presenting a bequest rule to the first child and presenting it to the second child become indifferent for the parents. The first or second child lives with the parents, and the one who does not live with them lives far away. In this case, the child who lives with the parents receives a larger part (or all) of the parental bequests. As the total amount of bequests increases further and exceeds another certain level, the parents present a bequest rule to the first child. The first child lives with the parents and the second child lives far away. In this case, the first child receives a larger part of the bequests.

The birth order, which implies first children choosing their location first, plays an important role in obtaining the above results. When the total amount of parental bequests is small, it gives rise to the differences in the level of reservation utility between the first and second children, leading to the result that the second child lives closer to the parents, as in Konrad *et al*. On the other hand, when the total amount of parental bequests is large, the fact that the parents can also influence the location of the second child, who has not yet decided where to live at the stage of their presenting a bequest rule to the first child, results in the first child living with the parents. The latter result is consistent with traditional Japanese family residential patterns, suggesting that they can be explained not only by cultural and social norms, but also by economic rationality.

Appendix

1. Proof of Proposition 1

We divide the region into $A\ (0 < s \le k^c)$ and $B\ (s \ge k^c)$ and derive s_0 that maximizes the second child's utility in each region[1]. Then, comparing the maximum solution in Region A with that in Region B, we obtain the overall maximum solution.

Lemma A-1

In Region A, we have $\hat{s} < s_0^A \le k^c$, where s_0^A is s_0 that maximizes the second child's utility in this region, under Assumption 1.

(Proof)

In Region A, we consider the following problem.

$$\underset{s}{Sup}\, u_k(Y_k(s)) + v_k(a(f_0) + a(s)) - c(a(s))$$
$$sub\ to\ 0 < s < k^c$$

We define $F(s)$ as

$$F(s) \equiv \partial U^k(s)/\partial s = u_k'(Y_k(s))Y_k'(s) + \left[v_k'(a(f_0) + a(s)) - c'(a(s))\right]a'(s).$$

From Assumption 1 and the assumption for $Y_k(s)$, we have

$$F(s) > 0 \quad \text{for } {}^\forall s \in [0,\ \hat{s}]. \tag{A1}$$

From the second order condition, we have

$$F'(s) < 0. \tag{A2}$$

Under (A1) and (A2), we have the following two cases:

(i) If $F(s) > 0$ for ${}^\forall s \in (\hat{s},\ k^c)$, then we have $s_0^A = \sup\{s \mid 0 < s < k^c\} = k^c$

---

[1] No strategic bequest motive arises for the parents, when $s_0 = 0$. Therefore, $s_0 = 0$ is excluded.

(ii) If $F(s)=0$ for ${}^{\exists}s \in (\hat{s},\ k^c)$, then we have $\hat{s} < s_0^A < k^c$

From (i) and (ii), we have *Lemma A-1.*

Lemma A-2

In Region B, $s_0^B = k^c$, where s_0^B is s_0 that maximizes the second child's utility in this region, under Assumption 1.

(Proof)

In Region B, we consider the following problem.

$$\underset{s}{Sup}\, u_k(Y_k(s)) + v_k(a(f_0)+a(s)) - c(a(s))$$
$$sub\ to\ s > k^c$$

In Region B, we have that, if $F(s)<0$ for ${}^{\forall}s \in (k^c,\ \infty)$, then we have

$s_0^B = \inf\{s \mid s > k^c\} = k^c$. □

From Lemma $A-1$ and Lemma $A-2$, we obtain Proposition 1.

Proof of Proposition 1

When we have $\hat{s} < s_0^A \le k^c$ in Region A and $s_0^B = k^c$ in Region B, we obtain $s_0 = s_0^A$ as the overall maximum solution, because $U^k(s_0^A) \ge U^k(k^c)$. On the other hand, when we have $s_0^A = k^c$ in Region A and $s_0^B = k^c$ in Region B, we obtain $s_0 = k^c$. □

2. The FOCs of the parents' problem.

The Lagrangian of the problem is as follows:

$$L(s,\beta_s,\lambda_s,\mu_s) = u_p(Y_p - b) + v_p(a(f_0) + a(s))$$
$$+\lambda_s\left[\left(u_k(Y_k(s)+\beta_s b) + v_k(a(f_0)+a(s)) - c(a(s)\right) - \left(u_k(Y_k(s_0)) + v_k(a(f_0)+a(s_0)) - c(a(s_0))\right)\right]$$
$$+\mu_s\left(1-\beta_s\right),\ given\ b,\ f_0.$$

From this, we obtain the FOCs of the parents' problem:

$$\frac{\partial L}{\partial s} = U_s^p + \lambda_s U_s^k \leq 0\,, \tag{A3}$$

$$s\frac{\partial L}{\partial s} = 0\,, \tag{A4}$$

$$\frac{\partial L}{\partial \beta_s} = \lambda_s U_{\beta_s}^k - \mu_s = 0\,, \tag{A5}$$

$$\frac{\partial L}{\partial \lambda_s} = U^k(s,\ \beta_s) - \bar{U}^s \geq 0\,, \tag{A6}$$

$$\lambda_s \frac{\partial L}{\partial \lambda_s} = 0\,, \tag{A7}$$

$$\frac{\partial L}{\partial \mu_s} = 1 - \beta_s \geq 0\,, \tag{A8}$$

$$\mu_s \frac{\partial L}{\partial \mu_s} = 0\,, \tag{A9}$$

where

$$U^p(s) = u_p(Y_p - b) + v_p(a(f_0) + a(s))\,,$$

$$U^k(s,\ \beta_s) = u_k(Y_k(s) + \beta_s b) + v_k(a(f_0) + a(s)) - c(a(s))\,,$$

$$U_s^p = \frac{\partial U^p}{\partial s} = v_p'(a(f_0) + a(s))a'(s) < 0\,,$$

$$U_s^k = \frac{\partial U^k}{\partial s} = u_k'(Y_k(s)) + \beta_s b)Y_k'(s) + \left[v_k'(a(f_0) + a(s)) - c'(a(s))\right]a'(s) > 0\,,$$

$$U_{\beta_s}^k = \frac{\partial U^k}{\partial \beta_s} = b u_k'(Y_k(s) + \beta_s b) > 0\,.$$

From (A3)-(A9), we have the equilibrium[2]

$s^* = s^*(f_0,\ b),\ \beta_s^* = \beta_s^*(f_0,\ b),\ \lambda_s^* = \lambda_s^*(f_0,\ b)$ and $\mu_s^* = \mu_s^*(f_0,\ b)$.

3. Proof of Proposition 2

Proof of Proposition 2 (i)

Since $s = 0$ does not satisfy the second child's participation constraint, (A3) is satisfied with equality:

$$U_s^p + \lambda_s U_s^k = 0. \quad \text{(A10)}$$

From (A10), noting that $U_s^p < 0$ and $U_s^k > 0$, we have

$$\lambda_s^* = -U_s^p / U_s^k > 0. \quad \text{(A11)}$$

Substituting (A11) into (A5) and noting $U_{\beta_s}^k > 0$, we have

$$\mu_s^* = -\frac{U_s^p}{U_s^k} U_{\beta_s}^k > 0.$$

From $\mu_s^* > 0$, (A8) and (A9), we obtain $\beta_s^* = 1$.

Also, (A6), (A7) and (A11) imply

$$u_k(Y_k(s) + \beta_s b) + v_k(a(f) + a(s)) - c(a(s)) = \overline{U}_s.$$

s^* is obtained from the participation constraint and $\beta_s^* = 1$. Since $s = 0$ does not satisfy the participation constraint, we have $s^* > 0$.

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[2] See Appendix 4 for the derivation of $ds^*(f_0,\ b)/df_0$ and $ds^*(f_0,\ b)/db$.

Proof of Proposition 2 (ii)

We will show that $s^* = 0,\ \hat{\beta}_s \le \beta_s^* \le 1,\ \lambda_s^* = 0$ and $\mu_s^* = 0$ satisfy (A3) - (A9). Substituting $\lambda_s^* = 0$ into (A3), $v_p' > 0$ and $a'(s) < 0$ imply

$$\frac{\partial L}{\partial s} = v_p'(a(f) + a(s^*))a_s'(s^*) < 0.$$

Thus we have

$s^* = 0$.

This satisfies (A4).

Substituting $\lambda_s^* = 0$ into (A5) yields

$$\mu_s^* = 0. \qquad \text{(A12)}$$

From (A8), (A9) and (A12) we have

$\beta_s^* \le 1$.

When $\lambda_s = 0$, (A7) is satisfied. Substituting $s^* = 0$ into (A6) yields

$$u_k(Y_k(0) + \beta_s^* b) + v_k(a(f_0) + a(0)) - c(a(0)) \ge \bar{U}_s. \qquad \text{(A13)}$$

When $\beta_s^* = \hat{\beta}_s$, (A13) is satisfied.

Differentiating the left-hand side of (A13) with respect to β_s yields

$$\partial\left[u_k(Y_k(0) + \beta_s^* b) + v_k(a(f) + a(0)) - c(a(0))\right] / \partial\beta_s = bu_k' > 0. \qquad \text{(A14)}$$

From (A14), $\beta_s^* \ge \hat{\beta}_s$ satisfies the participation constraint. □

4. The derivation of $ds^*(f_0,\ b)/df_0$ and $ds^*(f_0,\ b)/db$

When $\lambda_s^* > 0$, we have $\beta_s^* = 1$ and $s^* > 0$ in Proposition 2 (i). From (A3) and (A6), we have

$$\frac{\partial L}{\partial s} = v_p'(a(f_0) + a(s))a'(s) + \lambda_s\left[u_k'(Y_k(s)) + b)Y_k'(s) + \left[v_k'(a(f_0) + a(s)) - c'(a(s))\right]a'(s)\right] = 0 \tag{A15}$$

and

$$\frac{\partial L}{\partial \lambda_s} = \left(u_k(Y_k(s) + b) + v_k(a(f_0) + a(s)) - c(a(s)\right) - \left(u_k(Y_k(s_0)) + v_k(a(f_0) + a(s_0)) - c(a(s_0))\right) = 0. \tag{A16}$$

Differentiating (A15) and (A16) with respect to $s,\ \lambda_s,\ f_0,\ b$ yields

$$\begin{bmatrix} L_{ss} & L_{s\lambda_s} \\ L_{\lambda_s s} & 0 \end{bmatrix}\begin{bmatrix} ds \\ d\lambda_s \end{bmatrix} = \begin{bmatrix} -L_{sb} \\ -L_{\lambda_s b} \end{bmatrix} db + \begin{bmatrix} -L_{sf} \\ -L_{\lambda_s f} \end{bmatrix} df\ , \tag{A17}$$

where

$$L_{ss} = \left[v_p''(A) + \lambda_s v_k''(A)\right]\left(a'(s)\right)^2 + \left[v_p'(A) + \lambda_s(v_k'(A) - c'(a(s))\right]a''(s) + \lambda_s\left[u_k''\cdot(Y_k'(s))^2\right] < 0,$$

$$L_{s\lambda_s} = u_k'(Y_k(s) + b)Y_k'(s) + \left[(v_k'(A) - c'(a(s))\right]a'(s) > 0,$$

$$L_{sb} = \lambda_s\left[u_k''(Y_k(s) + b)\cdot Y_k'(s)\right] < 0,$$

$$L_{\lambda_s b} = u_k'(Y_k(s) + b) > 0,$$

$$L_{sf} = \left[v_p''(A) + \lambda_s v_k''(A)\right]a'(s)a'(f_0) < 0\ ,$$

$$L_{\lambda_s f} = \left[v_k'(a(f_0)+a(s^*)) - v_k'(a(f_0)+a(s_0)) \right] a'(f_0) > 0,$$

$$A = a(f_0) + a(s).$$

Substituting $db = 0$ into (A17) and using Cramer's rule yield

$$\frac{ds^*}{df_0} = \frac{\begin{vmatrix} -L_{sf} & L_{s\lambda_s} \\ -L_{\lambda_s f} & 0 \end{vmatrix}}{D} = -\frac{L_{\lambda_s f}}{L_{\lambda_s s}}$$

$$= -\frac{\left[v_k'(a(f_0)+a(s^*)) - v_k'(a(f_0)+a(s_0)) \right] a'(f_0)}{u_k'(Y_k(s))+b)Y_k'(s) + \left[v_k'(a(f_0)+a(s)) - c'(a(s)) \right] a'(s)} < 0,$$

where $D = -(L_{s\lambda_s})^2$.

Substituting $df = 0$ into (A17) and using Cramer's rule yield

$$\frac{ds^*}{db} = \frac{\begin{vmatrix} -L_{sb} & L_{s\lambda_s} \\ -L_{\lambda_s b} & 0 \end{vmatrix}}{D} = -\frac{L_{\lambda_s b}}{L_{\lambda_s s}}$$

$$= -\frac{u_k'(Y_k(s)+b)}{u_k'(Y_k(s))+b)Y_k'(s) + \left[v_k'(a(f_0)+a(s)) - c'(a(s)) \right] a'(s)} < 0.$$

When $\lambda_s = 0$, we have $s^* = 0$ and $\hat{\beta}_s \le \beta_s^* \le 1$ in Proposition 2 (ii). Therefore, we have

$$\left. \frac{ds^*}{db} \right|_{\hat{\beta}_s \le \beta_s^* \le 1} = 0$$

and

$$\left. \frac{ds^*}{df_0} \right|_{\hat{\beta}_s \le \beta_s^* \le 1} = 0.$$

5. The derivation of $\frac{\partial \overline{s}}{\partial \beta_f}$, $\frac{\partial \overline{s}}{\partial b}$ and $\frac{\partial \overline{s}}{\partial f}$

From $\overline{s}(f,\ \beta_f,\ b) = \arg\max_s u_k(Y_k(s) + (1-\beta_f)b) + v_k(a(f) + a(s)) - c(a(s))$, we have

$$u_k'(Y_k(\overline{s}) + (1-\beta_f)b) \cdot Y_k'(\overline{s}) + [v_k'(a(f) + a(\overline{s})) - c'(a(\overline{s}))]a'(\overline{s}) = 0. \tag{A18}$$

Differentiating (A18) with respect to $\overline{s}$, β_f, f and b yields

$$\left[u_k''(Y_k')^2 + v_k' \cdot a''(\overline{s}) + v_k'' \cdot (a'(\overline{s}))^2\right] ds = -bu_k''Y_k' d\beta_f + (1-\beta_f)u_k''Y_k' db + v_k'' \cdot (a'(\overline{s}))^2 df \tag{A19}$$

From (A19), we have

$$\frac{\partial \overline{s}}{\partial \beta_f} = \frac{bu_k''Y_k'}{V_{ss}} > 0,$$

$$\frac{\partial \overline{s}}{\partial b} = -\frac{(1-\beta_f)u_k''Y_k'}{V_{ss}} < 0,$$

$$\frac{\partial \overline{s}}{\partial f} = -\frac{v_k'' \cdot (a'(s))^2}{V_{ss}} < 0,$$

where $V_{ss} \equiv u_k''(Y_k')^2 + v_k' \cdot a''(s) + v_k'' \cdot (a'(s))^2 < 0$.

6. Proof of $s^*(f,\ b) < \overline{s}(f,\ \beta_f,\ b)$

The FOC for $\overline{s}$ and the FOC for the interior solution of s^* (when $\lambda_s > 0$) are respectively,

$$u_k'(Y_k(\overline{s}) + (1-\beta_f)b) \cdot Y_k'(\overline{s}) + [v_k'(a(f) + a(\overline{s})) - c'(a(\overline{s}))]a'(\overline{s}) = 0, \tag{A20}$$

$$\begin{aligned} & v_p'(a(f) + a(s^*))a'(s^*) \\ & + \lambda_s \left[u_k'(Y_k(s^*)) + (1-\beta_f)b)Y_k'(s^*) + \left[v_k'(a(f) + a(s^*)) - c'(a(s^*))\right]a'(s^*)\right] = 0. \end{aligned} \tag{A21}$$

We define $F(s)$ as

$$F(s) \equiv u_k'(Y_k(s) + (1-\beta_f)b) \cdot Y_k'(s) + [v_k'(a(f) + a(s)) - c'(a(s))]a'(s). \tag{A22}$$

From (A20), we have

$$F(\overline{s}) = 0. \tag{A23}$$

From (A21), we have

$$F(s^*) = \frac{-v_p'(a(f)+a(s^*))a'(s^*)}{\lambda_s}.$$

$v_p'(a(f)+a(s^*))a'(s^*) < 0$ and $\lambda_s > 0$ imply

$$F(s^*) > 0. \tag{A24}$$

From (A23) and (A24), we have $s^* < \overline{s}$, because the second-order condition $F' < 0$ holds. When $\lambda_s = 0$, we have $s^* = 0$. Therefore, we have $s^* < \overline{s}$. □

7. Condition for $a'(f) + a'(s^*)s_f^* < 0$

From Appendix 4 (the derivation of ds^*/df), we have

$$ds^*/df = -\frac{\left[v_k'(a(f)+a(s^*)) - v_k'(a(f)+a(s_0))\right]a'(f)}{u_k'(Y_k(s)) + \beta_s b)Y_k'(s) + \left[v_k'(a(f)+a(s)) - c'(a(s))\right]a'(s)}. \tag{A25}$$

Substituting (A25) into $a'(f) + a'(s^*)s_f^*$ yields

$$a'(f) + a'(s^*)s_f^* = a'(f)\left[1 - \frac{\left[v_k'(a(f)+a(s^*)) - v_k'(a(f)+a(s_0))\right]a'(s^*)}{u_k'(Y_k(s^*)) + \beta_s b)Y_k'(s^*) + \left[v_k'(a(f)+a(s^*)) - c'(a(s^*))\right]a'(s*)}\right]. \tag{A26}$$

From (A26), we have

$$(0 <)\frac{\left[v_k'(a(f)+a(s^*)) - v_k'(a(f)+a(s_0))\right]a'(s^*)}{u_k'(Y_k(s^*)) + \beta_s b)Y_k'(s^*) + \left[v_k'(a(f)+a(s^*)) - c'(a(s^*))\right]a'(s*)} < 1,$$

as the condition for $a'(f) + a'(s^*)s_f^* < 0$.

8. Proof of Proposition 3

We divide the region into $C\ (0 < f \le k^c)$ and $D\ (f \ge k^c)$, and derive f_0 that maximizes the first child's utility in each region[3]. Then, comparing the maximum solution in Region C with that in Region D, we obtain the overall maximum solution.

Lemma A-3

In Region C, we have $\hat{f} < f_0^C \le k^c$, where f_0^C is f_0 that maximizes the first child's utility in this region, under Assumption 2.

(Proof)

In Region C, we consider the following problem.

$$\underset{f}{Sup}\, u_k(Y_k(f)) + v_k(a(f) + a(s^*(f,\ b))) - c(a(f))$$
$$sub\ to\ 0 < f < k^c$$

We define $G(f)$ as

$$G(f) \equiv \partial U^k / \partial f = u_k'(Y_k(f))Y_k'(f) + v_k'(a(f) + a(s^*(f,\ b)))\left[a'(f) + a'(s^*)s_f^*\right] - c' \cdot a'(f).$$

From Assumption 2 and the assumption for $Y_k(f)$, we have

$$G(f) > 0 \quad \text{for } {}^{\forall}f \in [0, \hat{f}]. \tag{A27}$$

From the second order condition, we have

$$G'(f) < 0. \tag{A28}$$

Under (A27) and (A28), we have the following two cases:

(i) If $G(f) > 0$ for ${}^{\forall}f \in (\hat{f},\ k^c)$, then we have $f_0^C = \sup\{f \mid 0 < f < k^c\} = k^c$.

(ii) If $G(f) = 0$ for ${}^{\exists}f \in (\hat{f},\ k^c)$, then we have $\hat{f} < f_0^C < k^c$.

From (i) and (ii), we have *Lemma A-3.*

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[3] No strategic bequest motive arises for the parents, when $f_0 = 0$. Therefore, $f_0 = 0$ is excluded.

Lemma A-4

In Region D, $f_0^D = k^c$, where f_0^D is f_0 that maximizes the first child's utility in this region, under Assumption 2.

(Proof)

In Region D, we consider the following problem.

$$\underset{f}{Sup}\, u_k(Y_k(f)) + v_k(a(f) + a(s^*(f,\ b))) - c(a(f))$$

$$sub\ to\ f > k^c$$

In Region D, we have that, if $G(f) < 0$ for $^\forall f \in (k^c,\ \infty)$, then we have

$f_0^D = \inf\{f \mid f > k^c\} = k^c$. □

From Lemma $A-3$ and Lemma $A-4$, we obtain Proposition 3.

Proof of Proposition 3

When we have $\hat{f} < f_0^C \le k^c$ in Region C and $f_0^D = k^c$ in Region D, we obtain $f_0 = f_0^C$ as the overall maximum solution, because $U^k(f_0^C) \ge U^k(k^c)$. On the other hand, when we have $f_0^C = k^c$ in Region C and $f_0^D = k^c$ in Region D, we obtain $f_0 = k^c$. □

9. Regions I and II

When the parents offer the bequest rule to the first child, the location of the second child $\bar{s}$ is given by

$$\bar{s} = \arg\max_{\bar{s}} u_k(Y_k(\bar{s}) + (1-\beta_f)b) + v_k(a(f) + a(\bar{s})) - c(a(\bar{s})). \quad \text{(A29)}$$

The FOC for (A29) is given by

$$\partial U_k / \partial \bar{s} = u_k'(Y_k(\bar{s}) + (1-\beta_f)b) \cdot Y_k'(\bar{s}) + [v_k'(a(f) + a(\bar{s})) - c'(a(\bar{s}))]a'(\bar{s}) \ge 0. \quad \text{(A30)}$$

Since we consider case where $s_0 = k_c$, from Lemma $A-1$, we have

$$\partial U_k / \partial s = u_k'(Y_k(s) + (1-\beta_f)b) \cdot Y_k'(s) + [v_k'(a(f) + a(s)) - c'(a(s))]a'(s) > 0$$

for $^{\forall} s \in [0,\ k^c)$. (A31)

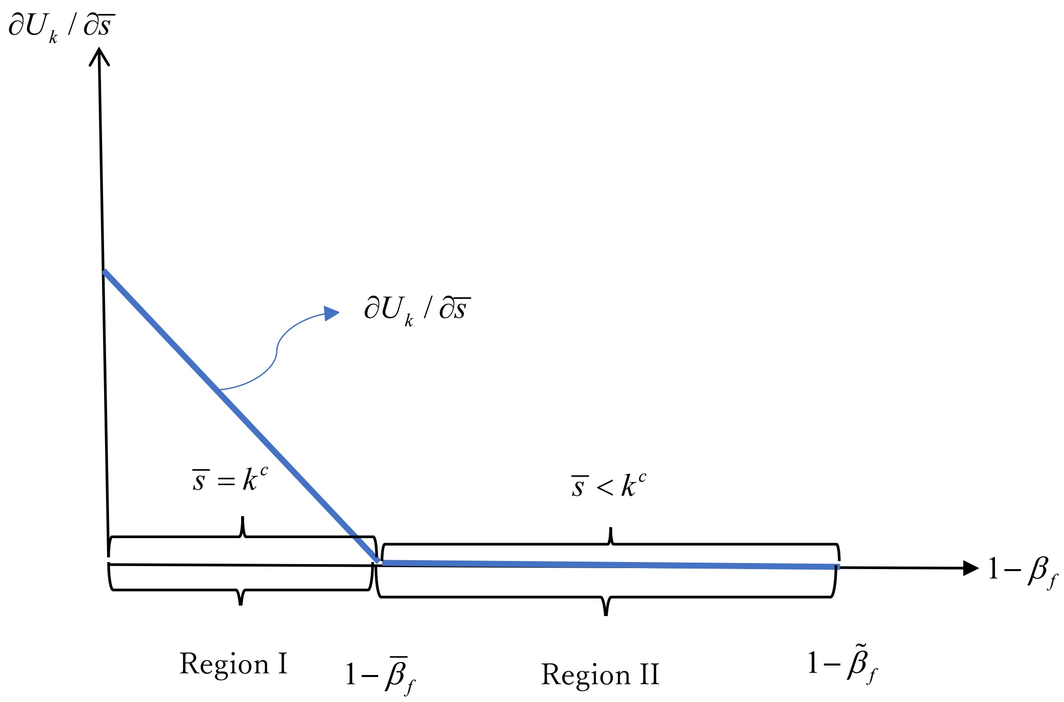
Noting that $Y_{k-}'(k^c) = Y_k'(\overline{s})$ for $^{\forall} s \in [0,\ k^c)$ [4], (A31) implies

$$\partial U_k / \partial s_0 = u_k'(Y_k(k^c)) \cdot Y_{k-}'(k^c) + [v_k'(a(f) + a(k^c)) - c'(a(k^c))]a'(k^c) > 0 .$$

Furthermore, differentiating $\partial U_k / \partial \overline{s}$ with respect to $1-\beta_f$ yields

$$\partial^2 U_k / \partial \overline{s} \partial (1-\beta_f) = b u_k''(Y_k(\overline{s}) + (1-\beta_f)b) \cdot Y_k'(\overline{s}) < 0 . \quad \text{(A32)}$$

Figure 15 Bequest distribution ratio to the second child and the marginal utility of $\overline{s}$



As shown in Figure 13, $\partial U_k / \partial \overline{s}$ decreases as $1-\beta_f$ increases from (A32), and

[4] We have $Y_{k-}'(k^c) = Y_k'(\overline{s})$ for $^{\forall} s \in [0,\ k^c)$ because $Y_k''(s) = 0$.

becomes zero at $1-\overline{\beta}_f$. If the bequest distribution ratio to the second child $1-\beta_f$ is less than $1-\overline{\beta}_f$, then the location of the second child is $\overline{s}=k_c$. If it is equal to or greater than $1-\overline{\beta}_f$, then the location of the second child is $\overline{s}\le k_c$.

10. Conditions for Assumption 3 to be satisfied

We show that, if $|a'(f)|\gg|a'(\overline{s})|$, $u_k''(Y_k')^2+(v'-c')a''(\overline{s})<0 \Leftrightarrow a'(f)+a'(\overline{s})\overline{s}_f<0$ and c' is sufficiently small, then $|MRS^K|<|MRS^P|$ ($MRS^K>MRS^P$) in the following.

We examine the relative magnitude between $|MRS^P|$ and $|MRS^K|$ in Region II.

$$
\begin{aligned}
-MRS^P+MRS^K &= \frac{U_f^P}{U_{\beta_f}^P}-\frac{U_f^K}{U_{\beta_f}^K}=\frac{U_f^P U_{\beta_f}^K-U_{\beta_f}^P U_f^K}{U_{\beta_f}^P U_{\beta_f}^K} \\
&= \frac{1}{U_{\beta_f}^P U_{\beta_f}^K}\Big(\left\{v_p'\cdot\left[a'(f)+a'(\overline{s})\overline{s}_f\right]\right\}\left\{bu_k'+v_k'\cdot a'(\overline{s})\overline{s}_{\beta_f}\right\} \\
&\qquad -\left\{v_p'\cdot a'(\overline{s})\overline{s}_{\beta_f}\right\}\left\{u_k'\cdot Y_k'(f)+v_k'\cdot\left(a'(f)+a'(\overline{s})\overline{s}_f\right)-c'\cdot a'(f)\right\}\Big) \\
&= \frac{v_p'}{U_{\beta_f}^P U_{\beta_f}^K}\left\{bu_k'a'(f)+\left[bu_k'\overline{s}_f-\left(u_k'\cdot Y_k'(f)-c'\cdot a'(f)\right)\overline{s}_{\beta_f}\right]a'(\overline{s})\right\}
\end{aligned}
\tag{A33}
$$

Substituting $\partial\overline{s}/\partial f=-v_k''a'(s)a'(f)/V_{ss}<0$ and $\partial\overline{s}/\partial\beta_f=bu_k''Y_k'/V_{ss}>0$ into the rightmost side of (A33) yields

$$
\begin{aligned}
&-MRS^P+MRS^K \\
&= \frac{v_p'}{U_{\beta_f}^P U_{\beta_f}^K}\left\{bu_k'a'(f)+\left[\frac{-bu_k'v_k''(a'(\overline{s}))^2a'(f)}{V_{ss}}+\frac{-bu_k'u_k''(Y_k'(f))^2a'(\overline{s})}{V_{ss}}+\frac{bu_k''Y_k'(f)c'a'(\overline{s})a'(f)}{V_{ss}}\right]\right\} \\
&= \frac{bv_p'}{U_{\beta_f}^P U_{\beta_f}^K}\left\{\left(u_k'-\frac{u_k'v_k''(a'(\overline{s}))^2}{V_{ss}}\right)a'(f)-\frac{u_k'u_k''(Y_k'(f))^2}{V_{ss}}a'(\overline{s})+\frac{u_k''Y_k'(f)a'(\overline{s})a'(f)}{V_{ss}}c'\right\},
\end{aligned}
\tag{A34}
$$

where $V_{ss}=u_k''(Y_k')^2+(v_k'-c')a''(\overline{s})+v_k''(a'(\overline{s}))^2<0$.

The first term in braces on the rightmost side of (A34) is written as follows.

$$\left(u_k' - \frac{u_k' v_k''(a'(\bar{s}))^2}{V_{ss}} \right) a'(f) = \frac{u_k' \left[u_k''(Y_k')^2 + (v' - c')a''(\bar{s}) \right]}{V_{ss}} a'(f) \tag{A35}$$

From $u_k''(Y_k')^2 + (v' - c')a''(\bar{s}) < 0 \ (\Leftrightarrow a'(f) + a'(\bar{s})\bar{s}_f < 0)$, the sign of (A35) is negative.

Hence, the first term is negative, the second term is positive and the third term is positive in braces on the rightmost side of (A34).

If $|a'(f)| \gg |a'(\bar{s})|$ and c' is sufficiently small[5], then we have

$$\left(u_k' - \frac{u_k' v_k''(a'(\bar{s}))^2}{V_{ss}} \right) a'(f) - \frac{u_k' u_k''(Y_k'(f))^2}{V_{ss}} a'(\bar{s}) + \frac{u_k'' Y_k'(f) a'(\bar{s}) a'(f)}{V_{ss}} c' < 0 . \tag{A36}$$

Therefore, (A36), $U^P_{\beta_f} < 0$ and $U^K_{\beta_f} > 0$ imply that the sign of (A33) and (A34) is positive. That is, the sign of $-MRS^P + MRS^K$ is positive.

Note that since $\bar{s}_f = \bar{s}_{\beta_f} = 0$ in Region I, the sign of (A33) is always positive.

11. Maximization problem and the Kuhn-Tucker conditions in Regions I and II

Maximization problem in Region I

In Region I, the parents choose a bequest rule which maximizes the parent's utility under the first child's participation constraint with $\bar{s} = k^c$. That is, the parents' problem is as follows:

$$\underset{f,\,\beta_f}{Max}\, u_p(Y_p - b) + v_p(a(f) + a(k^c)),$$

$$\begin{aligned} \textit{Sub to} \quad & U_k(f,\, \beta_f,\, b) \equiv u_k(Y_k(f) + \beta_f b) + v_k(a(f) + a(k^c) - c(a(f)) \\ & \geq u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\, b))) - c(a(f_0)), \\ & \bar{\beta}_f < \beta_f \leq 1, \ f \geq 0, \end{aligned} \tag{A37}$$

given b.

The Lagrangian of (A37) is given by

[5] $c'(= c'(a(f)))$ is a constant that does not depend on the size of $a(f)$ from the assumption of $c''(a(f)) = 0$.

$$L(f,\ \beta_f,\ \hat{\lambda}_f,\ \mu_f) = u_p(Y_p - b) + v_p(a(f) + a(k^c))$$
$$+\hat{\lambda}_f[\left(u_k(Y_k(f) + \beta_f b) + v_k(a(f) + a(k^c) - c(a(f))\right)$$
$$-\left(u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\ b)) - c(a(f_0))\right)] + \mu_f\left(1 - \beta_f\right).$$

The FOCs of (A37) are as follows:

$$\frac{\partial L}{\partial f} = U_f^P + \lambda_f U_f^K \le 0, \qquad \text{(A38)}$$

$$f\frac{\partial L}{\partial f} = 0, \qquad \text{(A39)}$$

$$\frac{\partial L}{\partial \beta_f} = \lambda_f U_{\beta_f}^K - \mu_f = 0, \qquad \text{(A40)}$$

$$\frac{\partial L}{\partial \lambda_f} = U^K(f,\ \beta_f) - \bar{U}^K \ge 0, \qquad \text{(A41)}$$

$$\lambda_f\frac{\partial L}{\partial \lambda_f} = 0, \qquad \text{(A42)}$$

$$\frac{\partial L}{\partial \mu_f} = 1 - \beta_f \ge 0, \qquad \text{(A43)}$$

$$\mu_f\frac{\partial L}{\partial \mu_f} = 0, \qquad \text{(A44)}$$

where

$$U^P(f) = u_p(Y_p - b) + v_p(a(f) + a(k^c)),$$

$$U^K(f,\ \beta_f) = u_k(Y_k(f) + \beta_f b) + v_k(a(f) + a(k^c) - c(a(f)),$$

$$U_f^P = \frac{\partial U^p}{\partial f} = v_p'(a(f) + a(k^c))a'(f) < 0,$$

$$U_f^K = \frac{\partial U^K}{\partial f} = u_k'(Y_k(f) + \beta_f b)Y_k'(f) + \left[v_k'(a(f) + a(k^c)) - c'\right]\cdot a'(f)) > 0,$$

$$U_{\beta_f}^K = \frac{\partial U^K}{\partial \beta_f} = b\cdot u_k'(Y_k(f) + \beta_f b) > 0,$$

$$\bar{U}^K = u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\ b))) - c(a(f_0)).$$

(A38)-(A44) yield $f^* = f^*(b)$, $\beta_f^* = \beta_f^*(b)$, $\hat{\lambda}_f^* = \hat{\lambda}_f^*(b)$ and $\mu_f^* = \mu_f^*(b)$.

Maximization problem in Region II

In Region II, the parents choose their bequest rule which maximizes the parents' utility under the first child's participation constraint with $\bar{s} < k^c$. That is, the parents' problem is as follows:

$$\underset{f,\,\beta_f}{Max}\, u_p(Y_p - b) + v_p(a(f) + a(\bar{s}(f,\, \beta_f,\, b)),$$

$$\textit{Sub to}\quad \begin{aligned} &U_k(f,\, \beta_f,\, b) \equiv u_k(Y_k(f) + \beta_f b) + v_k(a(f) + a(\bar{s}(f,\, \beta_f,\, b)) - c(a(f)) \\ &\geq u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\, b))) - c(a(f_0)), \end{aligned} \qquad \text{(A45)}$$

$$\tilde{\beta}_f < \beta_f \leq \bar{\beta}_f,\; f \geq 0,$$

given b.

The Lagrangian of (A45) is given by

$$L(f,\, \beta_f,\, \lambda_f,\, \sigma_f) = u_p(Y_p - b) + v_p(a(f) + a(\bar{s}(f,\, \beta_f,\, b)))$$
$$+\lambda_f[\left(u_k(Y_k(f) + \beta_f b) + v_k(a(f) + a(\bar{s}(f,\, \beta_f,\, b))) - c(a(f)\right)$$
$$-\left(u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*(f_0,\, b)) - c(a(f_0))\right)] + \sigma_f(\beta_f - \bar{\beta}_f).$$

The FOCs of (A45) are as follows:

$$\frac{\partial L}{\partial f} = U_f^P + \lambda_f U_f^K \leq 0, \qquad \text{(A46)}$$

$$f\frac{\partial L}{\partial f} = 0, \qquad \text{(A47)}$$

$$\frac{\partial L}{\partial \beta_f} = U_{\beta_f}^P + \lambda_f U_{\beta_f}^K - \sigma_f = 0, \qquad \text{(A48)}$$

$$\frac{\partial L}{\partial \lambda_f} = U^K(f,\, \beta_f) - \bar{U}^K \geq 0, \qquad \text{(A49)}$$

$$\lambda_f \frac{\partial L}{\partial \lambda_f} = 0, \qquad \text{(A50)}$$

$$\frac{\partial L}{\partial \sigma_f} = \bar{\beta}_f - \beta_f \geq 0, \qquad \text{(A51)}$$

$$\sigma_f \frac{\partial L}{\partial \sigma_f} = 0, \qquad \text{(A52)}$$

where

$$U^P(f,\, \beta_f) = u_p(Y_p - b) + v_p(a(f) + a(\bar{s}(f,\, \beta_f,\, b)),$$

$$U^K(f,\, \beta_f) = u_k(Y_k(f) + \beta_f b) + v_k(a(f) + a(\bar{s}(f,\, \beta_f,\, b))) - c(a(f),$$

$$U_f^P = \frac{\partial U^p}{\partial f} = v_p'(a(f)+a(s))\left[a'(f)+a'(\overline{s})\overline{s}_f\right] < 0,$$

$$U_f^K = \frac{\partial U^K}{\partial f} = u_k'(Y_k(f)+\beta_f b)Y_k'(f)$$
$$+v_k'(a(f)+a(\overline{s}(f,\ \beta_f,\ b)))\cdot\left(a'(f)+a'(\overline{s})\overline{s}_f\right)-c'\cdot a'(f)) > 0,$$

$$U_{\beta_f}^P = \frac{\partial U^P}{\partial \beta_f} = v_p'(a(f)+a(\overline{s}))\cdot a'(\overline{s})\overline{s}_{\beta_f} < 0,$$

$$U_{\beta_f}^K = \frac{\partial U^K}{\partial \beta_f} = b\cdot u_k'(Y_k(f)+\beta_f b)+v_k'(a(f)+a(\overline{s}(f,\ \beta_f,\ b)))\cdot a'(\overline{s})\overline{s}_{\beta_f} > 0.$$

(A46)-(A52) yield $f^* = f^*(b)$, $\beta_f^* = \beta_f^*(b)$, $\lambda_f^* = \lambda_f^*(b)$ and $\sigma_f^* = \sigma_f^*(b)$.

12. The maximum utility for the parents in Regions I and II

Region I

In case (i), since $f=0$ does not satisfy the first child participation constraint, (A38) is satisfied with equality:

$$U_f^P + \lambda_f U_f^K = 0. \tag{A53}$$

From (A53), we have

$$\lambda_f = -\frac{U_f^P}{U_f^K}. \tag{A54}$$

Substituting (A54) into (A40) yields

$$\mu_f = -\frac{U_f^P}{U_f^K}U_{\beta_f}^K. \tag{A55}$$

Noting that $U_f^P < 0$, $U_f^K > 0$ and $U_{\beta_f}^K > 0$, (A55) implies $\mu_f > 0$. From $\mu_f > 0$, (A43) and (A44), we obtain $\beta_f^* = 1$. From (A54) we have $\lambda_f^* > 0$. From $\lambda_f^* > 0$, (A41) and (A42), we have $U^K(f,\ \beta_f) = \overline{U}^K$. f^* is obtained from $U^K(f,\ \beta_f) = \overline{U}^K$

with $\beta_f^* = 1$. In case (i), since $f = 0$ does not satisfy the first child's participation constraint, we have $f^* > 0$.

In case (ii), we will show that $f^* = 0,\ \hat{\beta}_f \le \beta_f^* \le 1,\ \lambda_f^* = 0$ and $\mu_f^* = 0$ satisfy (A38) -(A44). Substituting $\lambda_f^* = 0$ into (A38) yields $\partial L / \partial f = v_p'(a(f) + a(k^c))a'(f) < 0$. Thus we have $f^* = 0$. Also, substituting $\lambda_f^* = 0$ into (A40) yields

$$\mu_f^* = 0. \qquad \text{(A56)}$$

From (A56), (A43) and (A44), we have $\beta_f^* \le 1$.

Furthermore, when $\hat{\beta}_f < \beta_f^* \le 1$, we have $\lambda_f^* = 0$. Substituting $f^* = 0$ into (A41) yields

$$u_p(Y_k(0) + \beta_f^* b) + v_k(a(0) + a(k^c)) - c(a(0)) \ge \bar{U}_f. \qquad \text{(A57)}$$

When $\beta_f^* = \hat{\beta}_f$, (A57) is satisfied with equality:

$$u_p(Y_k(0) + \hat{\beta}_f b) + v_k(a(0) + a(k^c)) - c(a(0)) = \bar{U}_f. \qquad \text{(A58)}$$

Differentiating the left-hand side of (A57) with respect to β_f^* yields

$$\partial\left[u_k(Y_k(0) + \beta_f^* b) + v_k(a(f) + a(0)) - c(a(0))\right] / \partial\beta_f^* = bu_k' > 0 \qquad \text{(A59)}$$

From (A58) and (A59), $\beta_f^* \ge \hat{\beta}_f$ satisfies the participation constraints.

In case (iii), since $U^K(f,\ \beta_f) > \bar{U}^K$, we have $\lambda_f^* = 0$ from (A41) and (A42). Substituting $\lambda_f^* = 0$ into (A38) yields $\partial L / \partial f = v_p'(a(f) + a(k^c))a'(f) < 0$. Thus we have $f^* = 0$. Also, substituting $\lambda_f^* = 0$ into (A40) yields $\mu_f^* = 0$. From $\mu_f^* = 0$, (A43) and (A44), we have $\beta_f^* \le 1$. □

Region II

In cases (i) and (ii), since $f=0$ does not satisfy the first child's participation constraint, (A46) is satisfied with equality: $\lambda_f = -U_f^P / U_f^K > 0$. Substituting this into (A48) yields

$$\hat{U}_{\beta_f}^P + \left(-\frac{\hat{U}_f^P}{\hat{U}_f^k}\right)\hat{U}_{\beta_f}^K - \sigma_f = \hat{U}_{\beta_f}^P\left[1+\left(\frac{\hat{U}_f^P}{\hat{U}_{\beta_f}^P}\right)\left(-\frac{\hat{U}_{\beta_f}^K}{\hat{U}_f^k}\right)\right] - \sigma_f = \hat{U}_{\beta_f}^P\left[1+\frac{-MRS^P}{MRS^K}\right] - \sigma_f = 0. \tag{A60}$$

From (A60), we have

$$U_{\beta_f}^P\left[-MRS^P + MRS^K\right] - \sigma_f MRS^K = 0. \tag{A61}$$

From (A61), noting that $\hat{U}_{\beta_f}^P < 0$ and $MRS^K < 0$, $-MRS^P + MRS^K > 0$ implies $\sigma_f > 0$. Hence, from (A51) and (A52), we have $\beta_f^* = \overline{\beta}_f$. Also, from (A49), (A50) and $\lambda_f^* = -U_f^P / U_f^K > 0$, we have $U^K(f^*, \beta_f^*) = \overline{U}^K$. f^* is obtained from $U^K(f^*, \beta_f^*) = \overline{U}^K$ with $\beta_f^* = \overline{\beta}_f$. In cases (i) and (ii), since $f=0$ does not satisfy the first child's participation constraint, we have $f^* > 0$.

In case (iii), we will show that $f^* = 0,\ \beta_f^* = \hat{\beta}_f \le \overline{\beta}_f,\ \lambda_f^* > 0$ and $\sigma_f^* = 0$ satisfy (A46)-(A52). Substituting $\sigma_f^* = 0$ into (A52) yields $\beta_f^* \le \overline{\beta}_f$ from (A51). Also substituting $\sigma_f^* = 0$ into (A48) yields $\lambda_f^* = -U_{\beta_f}^P / U_{\beta_f}^K > 0$. Substituting this into (A46), we have

$$\frac{\partial L}{\partial f} = U_f^P + \left(-\frac{U_{\beta_f}^P}{U_{\beta_f}^K}\right)U_f^K = U_{\beta_f}^P\left[\left(\frac{U_f^P}{U_{\beta_f}^P}\right) + \left(-\frac{U_f^K}{U_{\beta_f}^K}\right)\right] = U_{\beta_f}^P\left[-MRS^P + MRS^K\right]. \tag{A62}$$

From (A62), noting that $U_{\beta_f}^P < 0$, $-MRS^P + MRS^K > 0$ implies $\partial L / \partial f < 0$. Thus we have $f^* = 0$ from (A47). Also, from (A49), (A50) and $\lambda_f^* > 0$, we have

$U^K(f^*,\ \beta_f^*) = \bar{U}^K$. β_f^* is obtained from $U^K(f^*,\ \beta_f^*) = \bar{U}^K$ with $f^* = 0$, and hence $\beta_f^* = \hat{\beta}_f$. □

13. Proof of Lemma 5

We show that, as b increases, the first child's location f^* offered by the parents approaches the parent's location.

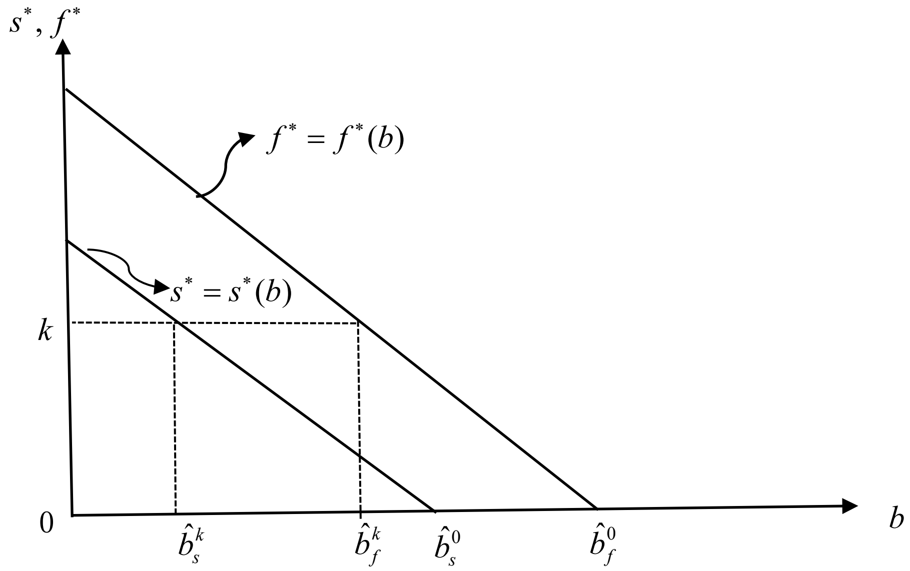
When the parents offer the bequest rule to the first child, noting that $\beta_f^* = 1$ and $\bar{s} = k^c$ ($\bar{s}_b = \bar{s}_f = 0$), the marginal utility of the parents with respect to b is

$$\begin{aligned} \frac{dU_p}{db} &= \frac{d}{db}\left\{u_p(Y_p - b) + v_p[a(f^*(b)) + a(\bar{s}(f^*(b),\ b))]\right\} \\ &= -u_p'(Y_p - b) + v_p'[a'(f^*)f^{*\prime}(b) + a'(\bar{s})(\bar{s}_f f^{*\prime}(b) + \bar{s}_b)] \qquad \text{(A63)} \\ &= -u_p'(Y_p - b) + v_p' \cdot a'(f^*)f^{*\prime}(b), \end{aligned}$$

where $f^{*\prime}(b) \equiv [df^*(b)/db]_{\beta_f^*=1}$. Suppose that $f^{*\prime}(b) \geq 0$. From (A63), we have $dU_p/db < 0$. From this, we have $b^* = 0$, which contradicts the assumption that $b^* > 0$. Therefore, we have $f^{*\prime}(b) < 0$. □

14. Proof of Lemma 6

Figure 16 Total bequest of parents and locations of first and second child



First, we examine the relative magnitude between $f^*(b)$ and $s^*(b)$ for $0 < b < \hat{b}_f^0$.

We define b which satisfies $U^s(s^*,\ \beta_s^*,\ b)\big|_{s^*=k,\ \beta_s^*=1} = \bar{U}^s$ as $\hat{b}_s^k$. We define $\hat{b}_f^k$ as b which satisfies $U^f(f^*,\ \beta_f^*,\ b)\big|_{f^*=k,\ \beta_f^*=1} = \bar{U}^f$. From these definitions, we have $f^*(\hat{b}_f^k) = s^*(\hat{b}_s^k) = k$ $(0 \le k < k^c)$ (See Figure 15).

When $b = \hat{b}_f^k$ and $s_0 = f_0 = k^c$, the participation constraint for the first child in equilibrium is

$$\begin{aligned} & u_k(Y_k(k) + \hat{b}_f^k) + v_k(a(k) + a(k^c)) - c(a(k)) \\ & = u_k(Y_k(k^c)) + v_k(a(k^c) + a(s^*(\hat{b}_f^k,\ k^c))) - c(a(k^c))(\equiv \bar{U}_f). \end{aligned} \tag{A64}$$

When $b = \hat{b}_s^k$ and $f_0 = s_0 = k^c$, the participation constraint for the second child in equilibrium is

$$u_k(Y_k(k)+\hat{b}_s^k)+v_k(a(k^c)+a(k))-c(a(k))$$
$$=u_k(Y_k(k^c))+v_k(2a(k^c))-c(a(k^c))(\equiv \bar{U}_s). \tag{A65}$$

Subtracting (A65) from (A64) yields

$$[u_k(Y_k(f^*)+\hat{b}_f^k)-u_k(Y_k(s^*)+\hat{b}_s^k)]=\bar{U}_f-\bar{U}_s. \tag{A66}$$

Using the mean value theorem on the left-hand side of (A66), we have

$$u_k'(A)(\hat{b}_f^k-\hat{b}_s^k)=\bar{U}_f-\bar{U}_s, \tag{A67}$$

where $A=\theta[Y_k(k)+\hat{b}_f^k]+(1-\theta)[Y_k(k)+\hat{b}_s^k]$. From (A67), we have

$$\hat{b}_f^k-\hat{b}_s^k=\frac{\bar{U}_f-\bar{U}_s}{u_k'(A)}. \tag{A68}$$

Since the right-hand side of (A66) is positive from footnote 1 in the text, (A68) implies

$$\hat{b}_f^k>\hat{b}_s^k \quad (0\le k<k^c). \tag{A69}$$

From (A69) and $s_b^*<0$, we have $s^*(\hat{b}_s^k)>s^*(\hat{b}_f^k)$. On the other hand, from $f^*(\hat{b}_f^k)=s^*(\hat{b}_s^k)=k$, we have $f^*(\hat{b}_f^k)>s^*(\hat{b}_f^k)$. Generally, we have $f^*(b)>s^*(b)$ for $0<b<\hat{b}_f^0$.

Next, we examine $f^*(b)$ and $s^*(b)$ for $b\ge\hat{b}_f^0$. Since we have $f^*(b)=0$ when $b=\hat{b}_f^0$, we also have $f^*(b)=0$ for $b>\hat{b}_f^0$. Similarly, since we have $s^*(b)=0$ when $b=\hat{b}_s^0$, we have $s^*(b)=0$ for $b>\hat{b}_s^0$. Since we have $\hat{b}_f^0>\hat{b}_s^0$ from (A69), we have $s^*(b)=0$ for $b\ge\hat{b}_f^0$. Therefore, we have $f^*(b)=s^*(b)=0$ for $b\ge\hat{b}_f^0$. □

15. Proof of Lemma 7

When $f^* = 0$, we have $\beta_f^* = \hat{\beta}_f^*$, $\lambda_f^* > 0$ and $\mu_f^* = \sigma_f^* = 0$ from (A38)-(A44) in Region I and (A46)-(A52) in Region II. Furthermore, from Lemma 6, we have $s^* = 0$ $(s_b^* = 0)$ for $b \ge b_f^0$. Therefore, the equilibrium is characterized by (A40) and (A41) with equality in Region I or (A48) and (A49) with equality in Region II.

$$\frac{\partial L}{\partial \hat{\beta}_f^*} = v_p'(a(f^*) + a(\overline{s})) \cdot a'(\overline{s})\overline{s}_{\hat{\beta}_f} + \lambda_f \left[b \cdot u_k'(Y_k(f^*) + \hat{\beta}_f^* b) + v_k'(a(f^*) + a(s(f,\ \hat{\beta}_f^*,\ b))) \cdot a'(\overline{s})\overline{s}_{\hat{\beta}_f} \right] = 0, \tag{A70}$$

$$\frac{\partial L}{\partial \lambda_f} = \left(u_k(Y_k(f^*) + \hat{\beta}_f^* b) + v_k(a(f^*) + a(s(f,\ \hat{\beta}_f^*,\ b))) - c(a(f^*)) \right) - \left(u_k(Y_k(f_0)) + v_k(a(f_0) + a(s^*)) - c(a(f_0)) \right) = 0. \tag{A71}$$

Differentiating (A70) and (A71) with respect to $\hat{\beta}_f^*$, λ_f and b yields.

$$\begin{bmatrix} L_{\hat{\beta}_f \hat{\beta}_f} & L_{\hat{\beta}_f \lambda_f} \\ L_{\lambda_f \hat{\beta}_f} & 0 \end{bmatrix} \begin{bmatrix} d\hat{\beta}_f^* \\ d\lambda_f \end{bmatrix} = \begin{bmatrix} -L_{\hat{\beta}_f b} \\ -L_{\lambda_f b} \end{bmatrix} db \tag{A72}$$

where

$$L_{\hat{\beta}_f \hat{\beta}_f} = v_p''(A) \cdot \left(a'(\overline{s})\overline{s}_{\hat{\beta}_f} \right)^2 + v_p'(A) \cdot a''(\overline{s})(\overline{s}_{\hat{\beta}_f})^2 + \lambda_f [b\hat{\beta}_f^* u_k''(C_f) + v_k'(A) a''(\overline{s})(\overline{s}_{\hat{\beta}_f})^2 + v_k''(A)(a'(\overline{s})(\overline{s}_{\hat{\beta}_f}))^2 < 0,$$

$$L_{\hat{\beta}_f \lambda_f} = b \cdot u_k'(C_f) + v_k'(A) \cdot a'(\overline{s})\overline{s}_{\hat{\beta}_f} > 0,$$

$$L_{\hat{\beta}_f b} = [v_p''(A) \cdot a'(\overline{s})^2 s_{\hat{\beta}_f} + v_p'(A) \cdot a''(\overline{s})\overline{s}_{\hat{\beta}_f}]\overline{s}_b + \lambda_f \{ u_k'(C_f) + u_k''(C_f) b^2 + [v_k''(A)(a'(\overline{s})^2 \overline{s}_{\hat{\beta}_f} + v_k'(A) a''(\overline{s})\overline{s}_{\hat{\beta}_f}]\overline{s}_b \},$$

$$L_{\lambda_f b} = \hat{\beta}_f^* u_k'(C_f) + v_k'(A)\cdot a'(\overline{s})\overline{s}_b > 0,$$

$$A \equiv a(f) + a(\overline{s}),\ C_f \equiv Y_k(f) + \hat{\beta}_f^* b.$$

From (A72), we have

$$\left.\frac{d\hat{\beta}_f^{\,*}}{db}\right|_{f^*=0} = \frac{\begin{vmatrix} -L_{\hat{\beta}_f b} & L_{\hat{\beta}_f \lambda_f} \\ -L_{\lambda_f b} & 0 \end{vmatrix}}{-(L_{\hat{\beta}_f \lambda_f})^2} = -\frac{L_{\lambda_f b}}{L_{\lambda_f \hat{\beta}_f}} < 0. \qquad \square$$

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